ALGEBRAIC CYCLES ON QUADRIC SECTIONS OF CUBICS IN $\mathbb{P}^4$ UNDER THE ACTION OF SYMPLECTOMORPHISMS

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Abstract. The Bloch-Beilinson's conjecture implies that the action of a symplectomorphism on the second Chow group of a $K3$-surface must be the identity. Generalizing the method developed in [20] we non-conjecturally prove the identity action of the Nikulin involution on the second Chow group for intersections of cubics and quadrics in $\mathbb{P}^4$, thus covering the $d = 3$ case in terms of [7]. Then we give explicit geometrical description of the invariant and anti-invariant components of the Prym variety associated to a smooth cubic 3-fold invariant under the above involution, and use it to show that the anti-invariant component vanishes when passing to 0-cycles on the generic fibre of an intersection with a pencil of invariant quadrics in $\mathbb{P}^4$.

1. Introduction

Let k be a subfield in $\mathbb{C}$, and let X be a $K3$-surface over k. Suppose we are given with a regular automorphism τ of X over k, such that its complexification $\tau_{\mathbb{C}}$ acts identically on the generator ω of the Hodge summand $H^{0,2}(X_{\mathbb{C}})$ in the second cohomology of the complex manifold $X_{\mathbb{C}}$. Then we say that τ is an algebraic symplectomorphism of the $K3$-surface X over k. The Bloch-Beilinson conjecture, [10], which is a global view in intersection theory, predicts that the action of τ on the second Chow group $CH^2(X)$ (with coefficients in $\mathbb{Q}$) is the identity, see [9].

Symplectomorphisms over $\mathbb{C}$ have order ≤ 8, see [16]. If τ is of order 2, i.e. τ is a Nikulin involution, then $\rho \geq 9$, where ρ is the rank of the Néron-Severi group $NS(X)$, see [7]. Assume that $\rho = 9$ and let L be a generator of the orthogonal complement of the lattice $E_8(-2)$ in $NS(X)$ whose self-intersection is $2d > 0$. Let Γ be a direct sum of $\mathbb{Z}L$ and $E_8(-2)$, if d is odd, or the unique even lattice containing $\mathbb{Z}L \oplus E_8(-2)$ as a sublattice of index 2, if d is even. For each Γ there exists a smooth projective $K3$-surface X with a Nikulin involution and $\rho = 9$, such that $NS(X) \simeq \Gamma$, and all such surfaces are parametrized by a coarse moduli space of dimension 11, see [7], Proposition 2.3.

The first results along this line had been obtained by C.Voisin in the breakthrough paper [20], where she used the power of the Hodge theory and subtle geometrical analysis to prove not only the Bloch's conjecture for finite quotients of quintics in the 3-dimensional projective space, but also the identity action of symplectomorphisms on quartics in $\mathbb{P}^3$ and intersections of 3 quadrics in $\mathbb{P}^5$, covering the cases $d = 2$ and $d = 4$ respectively. In [17] the identity action

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on $CH^2(X)$ was also proved for the case $d = 1$, and for $K3$-surfaces admitting elliptic pencils with sections.

The aim of the present work is to prove the identity action of the Nikulin involution for intersections of cubics and quadrics in $\mathbb{P}^4$, thus covering the $d = 3$ case in terms of the classification given in [7]. We show that the method in Voisin's paper, from both Hodge-theoretical and geometrical side, can be generalized an applied not only to pencils of surfaces in $\mathbb{P}^3$ but also to pencils of surfaces in a smooth projective threefold whose second Chow group is small.

But not merely that. The main insight in our computations here is that 0-cycles on a surface can and should be interpreted as codimension 2 algebraic cycles on their 3-dimensional spreads. Of course, this idea would actually be of much use to us only when the second Chow group of a suitable threefold is manageable. For example, in the case $d = 3$ the corresponding $K3$-surfaces are fibres of pencils on quadrics in $\mathbb{P}^4$ whose second Chow group is small. Equally, one can fibre cubics by quadrics and connect 0-cycles in the generic fibre with the second Chow group of cubics given in terms of their Prymians. It is then natural to ask how the Nikulin involution, which changes the sigh of 2 coordinates in $\mathbb{P}^4$, acts on codimension 2 algebraic cycles on cubic threefolds in $\mathbb{P}^4$. Our second aim is to give a precise geometrical description of such an action in terms of Prym varieties, and split the corresponding Prymian into invariant and anti-invariant parts. Then we use the obtained results in order to show that 0-cycles on the $K3$-intersection of a cubic and quadric in $\mathbb{P}^4$, supported on the intersection with an additional hypersurface of degree 3 or 2, vanish modulo rational equivalence. The latest result brings new information on rational deformations of points on $K3$-surfaces. In a sense, we want to use intersections of cubics and quadrics in $\mathbb{P}^4$ to embody the philosophy of spreads into concrete geometry, in order to show that the subtle conjectural phenomena on 0-dimensional algebraic cycles on surfaces may depend actually on very precise coordinate-like computations in the world of three-dimensional varieties.

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2. Terminology and notation

All Chow groups will be with coefficients in $\mathbb{Q}$, so that they will actually be $\mathbb{Q}$-vector spaces rather than abelain groups, but we shall keep the term "group" throughout the paper. For any smooth projective variety X over a field k, and any non-negative integer p, let

$$CH^p(X)$$

be the Chow group, with coefficients in $\mathbb{Q}$, of algebraic cycles of codimension p on X modulo rational equivalence relation, and let

$$A^p(X) = \{\alpha \in CH^p(X) \mid \alpha \text{ is algebraically equivalent to } 0 \text{ on } X\}$$

be a subgroup generated by cycles algebraically equivalent to zero. When needed to work with cycle classes integrally, we will be using the notation $CH^p_{\mathbb{Z}}(X)$, $A^p_{\mathbb{Z}}$, etc.

If X and Y are two smooth projective varieties over k, $d = \dim(X)$, let

$$CH^n(X,Y) = CH^{d+n}(X \times Y)$$

be the group of correspondences of degree n from X to Y. If

$$p : X \vdash Y \qquad \text{and} \qquad q : Y \vdash Z$$

are two correspondences of degree a and b respectively, their product is defined by the formula

$$q \circ p = p_{XZ*}(p_{XY}^*(p) \cdot p_{YZ}^*(q)) \; ,$$

where p_{XY}, p_{YZ} and p_{XZ} are the projections from $X \times Y \times Z$ onto the corresponding factors.

Having introduced correspondences one can define a category $\mathscr{CM}(k)$ of Chow motives over k in the standard way. Its objects are triples (X,p,m), where X is a smooth projective variety over k, p is a projector on X, i.e. a correspondence $p : X \vdash X$, such that $p^2 = p$, and m is an integer. The Hom-group from $M = (X,p,m)$ to $N = (Y,q,n)$ is nothing but the group

$$\mathrm{Hom}_{\mathscr{CM}(k)}(M,N) = q \circ CH^{n-m}(X,Y) \circ p \; ,$$

and the composition of morphisms in $\mathscr{CM}(k)$ is the above composition of correspondences over k. If $f : X \to Y$ is a regular morphism over k then let Γ_f^{t} be the transpose of its graph, which is an element in $CH^0(Y,X)$. Defining $M(X) = (X, \Delta_X, 0)$ and $M(f) = \Gamma_f^{\mathrm{t}} \in CH^0(Y,X)$ we obtain a contravariant functor $M(-)$ from the category of smooth projective varieties over k in to the category of Chow motives $\mathscr{CM}(k)$.

It is important to emphasis that $\mathscr{CM}(k)$ is tensor rigid with a product given by the formula

$$(X,p,m) \otimes (Y,q,n) = (X \times Y, p \otimes q, m+n) \; ,$$

where $p \otimes q$ is just a fibred product of two correspondences, so that the functor $M(-)$ is tensor. Moreover, $\mathscr{CM}(k)$ is rigid and obviously pseudo-abelian, so that we have wedge $\wedge^n$ and symmetric Sym^n power endofunctors in $\mathscr{CM}(k)$, which allow to define finite-dimensional motives in the sense of [12]. The substantial thing is the Kimura's theorem saying that if f is a numerically trivial endomorphism of a finite-dimensional motive M then f is nilpotent in the associative algebra $\mathrm{End}(M)$, loc.cit.

Further details on Chow motives and their finite-dimensionality can be found in [19] and [12].

Throughout the paper we will work basically over $\mathbb{C}$, but sometimes we will be also working over subfields k in $\mathbb{C}$, or even over the function fields of algebraic varieties defined over k or $\mathbb{C}$. This is why, by default, the cohomology groups $H^*(-,A)$ will be the Betti cohomology with coefficients $A = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$, but working over k or function fields, we will also get involved étale cohomology groups, which will be mentioned separately.

Let then

$$CH^p_{\mathbb{Z}}(X)_{\mathrm{hom}}$$

be the kernel of the cycle class homomorphism

$$cl : CH^p_{\mathbb{Z}}(X) \longrightarrow H^{2p}(X,\mathbb{Z}) \; .$$

Certainly, we also have a rational coefficient cycle class map

$$cl : CH^p(X) \longrightarrow H^{2p}(X,\mathbb{Q}) \ ,$$

whose kernel is $CH^p(X)_{\mathrm{hom}}$.

Each group $H^i(X,\mathbb{Q})$ carries a pure Hodge structure. Let F^p be the corresponding decreasing Hodge filtration on the complexified vector space $H^i(X,\mathbb{C})$, compatible with the complex conjugate filtration $\bar{F}^p$, and let $H^{p,q}(X)$ be the adjoint quotient $(F^p/F^{p+1})H^{p+q}(X,\mathbb{C})$. For any two irreducible smooth projective X and Y, and a correspondence $\alpha \in CH^m(X,Y)$, one has a homomorphism

$$\alpha_* : H^i(X,\mathbb{Q}) \longrightarrow H^{i+2m}(Y,\mathbb{Q}) \ ,$$

shifting the Hodge filtration by the formula

$$\alpha_*(F^pH^i(X,\mathbb{C})) \subset F^{p+m}H^{i+2m}(Y,\mathbb{C}) \ .$$

Then α indices a homomorphism on the Hodge components,

$$\alpha_* : H^{p,q}(X) \longrightarrow H^{p+m,q+m}(Y) \ .$$

Let

$$J^{2p-1}(X) = H^{2p-1}(X,\mathbb{C})/(\mathrm{Im}(H^{2p-1}(X,\mathbb{Z})) + F^pH^{2p-1}(X,\mathbb{C})) \ ,$$

be the p-th intermediate Jacobian of X. Here $\mathrm{Im}(H^{2p-1}(X,\mathbb{Z}))$ is the image of the natural map

$$H^{2p-1}(X,\mathbb{Z}) \longrightarrow H^{2p-1}(X,\mathbb{C}) \ ,$$

see [21, Section 12.1.1]. Then we have the Abel-Jacobi homomorphism

$$AJ : CH^p_{\mathbb{Z}}(X)_{\mathrm{hom}} \longrightarrow J^{2p-1}(X) \ ,$$

which will play an important rule in what follows, see Section 12.1.2 in loc.cit. We will also need the group

$$J^{2p-1}(X)_{\mathbb{Q}} = J^{2p-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

which can be described as a quotient

$$J^{2p-1}(X)_{\mathbb{Q}} = H^{2p-1}(X,\mathbb{C})/((H^{2p-1}(X,\mathbb{Q}) + F^pH^{2p-1}(X,\mathbb{C})) \ .$$

Extending coefficients from $\mathbb{Z}$ to $\mathbb{Q}$, we obtain a rational version of the Abel-Jacobi homomorphism

$$AJ : CH^p(X)_{\mathrm{hom}} \longrightarrow J^{2p-1}(X)_{\mathbb{Q}} \ .$$

The action of correspondences on Chow groups is compatible with their actions on Hodge structures via the cycle class map and the Abel-Jacobi map.

Let $J^{2p-1}_{\mathrm{alg}}(X)$ be the largest complex subtorus of $J^{2p-1}(X)$ whose tangent space is contained in $H^{p-1,p}(X)$. Then $J^{2p-1}_{\mathrm{alg}}(X)$ is an abelian variety over $\mathbb{C}$, see [21, Section 12.2.2]. As algebraic equivalence is finer than the homological one,

$$cl(CH^p(X)_{\mathrm{alg}}) = 0 \ .$$

It is well known that $AJ(A^p_{\mathbb{Z}}(X))$ is a subset in $J^{2p-1}_{\mathrm{alg}}(X)$ and, respectively, $AJ(A^p(X))$ is a subset in $J^{2p-1}_{\mathrm{alg}}(X)_{\mathbb{Q}}$, loc.cit.

Since $H^*(-,\mathbb{Q})$ is a Weil cohomology theory, i.e. it possesses a $\cup$-pairing, Poincaré duality etc, each $\mathbb{Q}$-vector subspace $V \subset H^d(X,\mathbb{Q})$ has a natural complement $V^\perp$, orthogonal to V with respect to the $\cup$ pairing in $H^d(X,\mathbb{Q})$, and we have a splitting

$$H^i(X,\mathbb{Q}) = V \oplus V^\perp\ .$$

For example, let X be a surface, i.e. $d = 2$, and let $NS(X)$ be the Néron-Severy group of X. Let then $H^2(X,\mathbb{Q})_{\rm alg}$ be the image of the cycle class map from $NS(X)\otimes\mathbb{Q}$ to $H^2(X,\mathbb{Q})$, and let $H^2(X,\mathbb{Q})_{\rm tr}$ be the orthogonal complement $(H^2(X,\mathbb{Q})_{\rm alg})^\perp$, so that we have a splitting of $\mathbb{Q}$-vector spaces

$$H^2(X,\mathbb{Q}) = H^2(X,\mathbb{Q})_{\rm alg} \oplus H^2(X,\mathbb{Q})_{\rm tr}\ .$$

We will say that $H^2(X,\mathbb{Q})$ is algebraic if $H^2(X,\mathbb{Q})_{\rm tr}$ is trivial. This is equivalent to say that $p_g = 0$, where $p_g = \dim H^2(X,\mathscr{O}_X)$ is the geometric genus of the surface X.

Another splitting of cohomology arises from a group action. Let X be a smooth projective variety of dimension d and assume that a finite group G acts by regular automorphisms on X. Then this action induces also an action of G on $H^p(X,\mathbb{Q})$ for any p. Let $H^p(X,\mathbb{Q})^G$ be the $\mathbb{Q}$-vector space of G-invariant elements in $H^p(X,\mathbb{Q})$. Let $H^{2d-p}(X,\mathbb{Q})^\sharp$ be the orthogonal complement $(H^p(X,\mathbb{Q})^G)^\perp$ to $H^p(X,\mathbb{Q})^G)$ with respect to the pairing between $H^p(X,\mathbb{C})$ and $H^{2d-p}(X,\mathbb{C})$. Then we have a splitting

$$H^p(X,\mathbb{Q}) = H^p(X,\mathbb{Q})^G \oplus H^p(X,\mathbb{Q})^\sharp\ .$$

It is worth to say that this decomposition is a decomposition in the category of Hodge structures $\mathscr{HS}$. Namely, let

$$H^p(X,\mathbb{C}) = H^{p,0}(X) \oplus H^{p-1,1}(X) \oplus \cdots \oplus H^{0,p}(X)$$

be the Hodge decomposition of $H^p(X,\mathbb{C})$. For each pair (i,j), where $i,j \in \{0,1,\dots,p\}$ and $i+j=p$, let $H^{i,j}(X)^G$ be a G-invariant subspace in $H^{i,j}(X)$, and let $H^{d-i,d-j}(X)^\sharp$ be the orthogonal complement to $H^{i,j}(X)^G$ with respect to the pairing between $H^p(X,\mathbb{C})$ and $H^{2d-p}(X,\mathbb{C})$. Then we have the following splittings

$$H^{i,j}(X) = H^{i,j}(X)^G \oplus H^{i,j}(X)^\sharp\ ,$$
$$H^p(X,\mathbb{C})^G = H^{p,0}(X,)^G \oplus H^{p-1,1}(X)^G \oplus \cdots \oplus H^{0,p}(X)^G$$

and

$$H^{2d-p}(X,\mathbb{C})^\sharp = H^{2d-p,0}(X,)^\sharp \oplus H^{2d-p-1,1}(X)^\sharp \oplus \cdots \oplus H^{0,2d-p}(X)^\sharp\ .$$

This gives rise to the invariant and anti-invariant parts in the intermediate Jacobian,

$$J^{2p-1}(X)^G \qquad \text{and} \qquad J^{2p-1}(X)^\sharp\ .$$

The fact that the action of the group G is compatible with the duality between $H^{i,j}(X)$ and $H^{d-j,d-i}(X)$ has another important implication which is better visible in dimension 2. So let again X is a surface, then $H^{2,0}(X,)^G = 0$ if and only if $H^{0,2}(X,)^G = 0$, and $H^{2,0}(X,)^\sharp = 0$ if and only if $H^{0,2}(X,)^\sharp = 0$. Thus, if $H^{2,0}(X,)^G = 0$ then $H^2(X,\mathbb{C})^G = H^{1,1}(X)^G$, whence the $\mathbb{Q}$-vector space $H^2(X,\mathbb{Q})^G$ is algebraic, in the sense that any cohomology class in $H^2(X,\mathbb{Q})^G$ comes from the class in $NS(X)\otimes\mathbb{Q}$ via the cycle class map. Similarly, if $H^{2,0}(X,)^\sharp = 0$ then $H^2(X,\mathbb{C})^\sharp = H^{1,1}(X)^\sharp$, so that the $\mathbb{Q}$-vector space $H^2(X,\mathbb{Q})^\sharp$

is algebraic, i.e. a cohomology class in $H^2(X,\mathbb{Q})^\sharp$ comes from a certain class in the Neron-Severi group $NS(X)\otimes\mathbb{Q}$ via the cycle class map.

3. Motivic setup

In this section we outline the motivic aspects of the $K3$-symplectomorphism action. The main thing here is the so-called transcendental motive $t^2(X)$ of a smooth projective surface X over k, see [11]. To make things easier let $\bar{k} = k$ be an algebraically closed field of characteristic 0, and suppose that the irregularity of X is 0 too. Let ρ be the rank of the Néron-Severi group of X, and choose ρ divisors $D_1, \dots, D_\rho$, generating the algebraic part $H^2_{\mathrm{alg}}(X)$ in the second étale l-adic cohomology of X, where l is a prime, and choose also their Poincaré dual divisors $D'_1, \dots, D'_\rho$ on X. Fixing a closed point P_0 on X, one can define a transcendental second Murre projector as a correspondence

$$\pi_2^{\mathrm{tr}}(X) = \Delta_X - [P_0\times X] - \sum_{i=1}^{\rho}[D_i\times D'_i] - [X\times P_0]\ .$$

in $CH^0(X,X)$. Then

$$t^2(X) = (X, \pi_2^{\mathrm{tr}}(X), 0)$$

in the transcendental motive of X in the category $\mathscr{CM}(k)$.

For any algebraic scheme X over k and a field extension L/k let $X_L = X\times_{\mathrm{Spec}(k)}\mathrm{Spec}(L)$, and for any morphism $f : X \to Y$ of schemes let $f_L = f\times_{\mathrm{Spec}(k)}\mathrm{Spec}(L)$. Similarly, one can define the category of Chow motives $\mathscr{CM}(L)$ over L, and a tensor functor $\mathrm{res}_{L/k} : \mathscr{CM}(k) \to \mathscr{CM}(L)$.

Let now k be an algebraically closed subfield in $\mathbb{C}$, let X be a smooth projective $K3$-surface over k, and let

$$\omega \in H^{0,2}(X_{\mathbb{C}})$$

be a symplectic form on $X_{\mathbb{C}}$. Suppose, furthermore, that

$$G = \{\mathrm{id}, g, g^2, \dots, g^{n-1}\}$$

is a finite cyclic subgroup in the group $\mathrm{Aut}(X)$ of regular automorphisms of the surface X over k, such that

$$g^*_{\mathbb{C}}(\omega) = \omega\ .$$

Then we say that g is an algebraic symplectomorphism on the surface X. Our aim in this section is to show that if the Bloch-Beilinson conjecture[1] is true, then the induced action $g^*_{\mathbb{C}} : CH^2(X_{\mathbb{C}}) \to CH^2(X_{\mathbb{C}})$ on the codimension 2 Chow group is the identity. As all Chow groups in this paper are with coefficients in $\mathbb{Q}$, it would follow that the homomorphism $g^*_L : CH^2(X_L) \to CH^2(X_L)$ is the identity for any field subextension $k\subset L\subset\mathbb{C}$.

Let Y be the quotient X/G with a quotient map $f : X \to Y$, and let $Y' \to Y$ be a resolution of singularities of the surface Y arising from the points on X whose orbits under the action of G are shorter than n. Consider the following

[1] see [10] for the exposition of the Bloch-Beilinson paradigm

Cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in the category of schemes over k, see [7]. Then Y' and X' are both smooth projective $K3$-surfaces over k, loc.cit.

Since the above commutative square is Cartesian, any automorphism $a : X \to X$ gives rise to a uniquely defined automorphism $a' : X' \to X'$, so that we have a natural embedding of $\mathrm{Aut}(X)$ into $\mathrm{Aut}(X')$. Let then $G' = \{\mathrm{id}, g', g'^2, \dots, g'^{n-1}\}$ be the image of the group G under the embedding of $\mathrm{Aut}(X)$ into $\mathrm{Aut}(X')$.

Let Γ be the graph of the morphism $f' : X' \to Y'$ and let Γ^{t} be its transpose. Then

$$\Gamma^{\mathrm{t}}\Gamma = \sum_{i=0}^{n-1} (\mathrm{id}_{X'} \times (g')^i)^* \Delta_{X'} \ .$$

Let

$$p = \frac{1}{n}\,\Gamma^{\mathrm{t}}\Gamma \ ,$$

be the correspondence defining the embedding of the Chow motive $M(Y')$ into the Chow motive $M(X')$. The correspondence p is a projector, so that p induces a splitting

$$M(X') = M(Y') \oplus N \ ,$$

where N is a motive defined by the projector

$$q = \Delta_{X'} - p \ .$$

It is not hard to see that f' induces an isomorphism on the algebraic parts in the second Murre motive M^2. Then we also have a splitting

$$t^2(X') = t^2(Y') \oplus N \ ,$$

and

$$H^*(N) = 0 \ ,$$

where $H^*(-)$ stays for a Weil cohomology theory. The transcendental motives of X' and X are isomorphic,

$$t^2(X') \simeq t^2(X) \ ,$$

as a blowing up only paste exceptional curves which do not make any impact on the transcendental motive of X. The inclusion $t^2(Y') \hookrightarrow t^2(X')$, induced by p, is an isomorphism if and only if the above N vanishes in $\mathscr{CM}(k)$.

Lemma 1. *Let L/k be a field extension, where L is algebraically closed. The inclusion $t^2(Y') \hookrightarrow t^2(X')$ is an isomorphism in $\mathscr{CM}(k)$ if and only if the inclusion $t^2(Y'_L) \hookrightarrow t^2(X'_L)$ is an isomorphism in $\mathscr{CM}(L)$.*

Proof. Let

$$N_L = \mathrm{res}_{L/k}(N) \ .$$

As we work with Chow groups with coefficients in $\mathbb{Q}$, the motive N_L vanishes in $\mathscr{CM}(L)$ if and only if the motive N vanishes in $\mathscr{CM}(k)$. □

Let now

$$F = k(X)$$

be the function field of the surface X over k, let

$$\eta = \mathrm{Spec}(F)$$

be the generic point of the surface X over k, and let

$$\eta : \mathrm{Spec}(F) \longrightarrow X$$

be the corresponding morphism of schemes over k. Consider a pull-back

$$\Phi_X : CH^2(X \times X) \longrightarrow CH^2(X_F) ,$$

induced on the Chow groups by the morphism $\mathrm{id}_X \times \eta$, [11], and let

$$P_\eta = \Phi_X(\Delta_X)$$

be an F-rational point on the surface X_F, the so-called generic zero-cycle on X_F.

The following lemma is analogous to Corollary 1 in [17].

Lemma 2. *In the notation above, the following three conditions are equivalent:*

(i) the embedding $t^2(Y') \hookrightarrow t^2(X')$, induced by the morphism f', is an isomorphism;

(ii) the homomorphism $g_F^* : CH^2(X_F) \to CH^2(X_F)$ is the identity;

(iv) the homomorphism $g_F^* : CH^2(X_F) \to CH^2(X_F)$ does not move the class of the point P_η in $CH^2(X_F)$.

Proof.

$(i) \Rightarrow (ii)$

Let $\bar{F}$ be an algebraic closure of the function field F. By Lemma 1, the embedding $t^2(Y'_{\bar{F}}) \hookrightarrow t^2(X'_{\bar{F}})$ is an isomorphism. The motive $t^2(Y'_{\bar{F}})$ is nothing but "the fixed part" of the action of $g'_{\bar{F}}$ on $t^2(X'_{\bar{F}})$. If the embedding $t^2(Y'_{\bar{F}}) \hookrightarrow t^2(X'_{\bar{F}})$ is an isomorphism, i.e. the fixed motive $t^2(Y'_{\bar{F}})$ coincides with the whole $t^2(X'_{\bar{F}})$, then this action is the identity. As the image of the projector π_2^{tr}, determining the motive $t^2(X'_{\bar{F}})$, is exactly the Albanese kernel for $X'_{\bar{F}}$, it follows that the action of $g'_{\bar{F}}$ on the Albanese kernel of the $K3$-surface $X'_{\bar{F}}$ is the identity. Therefore, the action of $g'_{\bar{F}}$ on the whole Chow group $CH^2(X'_{\bar{F}})$ is the identity. Using the behaviour of Chow groups under blow ups, and that the action of g' on X' commutes with the action of g on X with respect to the blowing up morphism $X' \to X$, we deduce that the action of $g_{\bar{F}}$ on the Chow group $CH^2(X_{\bar{F}})$ is the identity. As we work with coefficients in $\mathbb{Q}$, it follows that the action of g_F on the Chow group $CH^2(X_F)$ is the identity.

$(ii) \Rightarrow (iii)$

Obvious.

$(iii) \Rightarrow (i)$

Let $F' = k(X')$ be the function field of the surface X' over k, let $\eta' = \mathrm{Spec}(F')$ be the generic point of the surface X', and let $P_{\eta'} = \Phi_{X'}(\Delta_{X'})$ be the corresponding generic zero-cycle on $X'_{F'}$. Since the morphism $X' \to X$ is just a

blowing up of a finite collection of points on a surface, the corresponding pull-back gives an isomorphism $CH^2(X) \simeq CH^2(X')$, and, respectively, an isomorphism $CH^2(X_F) \simeq CH^2(X'_F)$. As the function field F' is actually the same as the function field F, we have a canonical isomorphism $CH^2(X_F) \simeq CH^2(X'_{F'})$. Moreover, action of g' on X' is compatible with the action of g on X. It follows that, since g_F^* does not move the class of the point P_η in $CH^2(X_F)$ by the assumption in (iii), the homomorphism $(g'_{F'})^*$ does not move the class of the point $P_{\eta'}$ in $CH^2(X'_{F'})$. The square

$$\begin{tikzcd}
CH^2(X' \times X') \arrow[r, "\Phi_{X'}"] \arrow[d, "(\mathrm{id}_{X'} \times (g')^i)^*"'] & CH^2(X'_{F'}) \arrow[d, "((g'_{F'})^i)^*"] \\
CH^2(X' \times X') \arrow[r, "\Phi_{X'}"] & CH^2(X'_{F'})
\end{tikzcd}$$

is commutative for any i. As $((g'_{F'})^i)^*(P_{\eta'}) = P_{\eta'}$, we have that

$$\Phi_{X'}(\mathrm{id}_{X'} \times (g')^i)^*(\Delta_{X'}) = ((g'_{F'})^i)^*\Phi_{X'}(\Delta_{X'}) = ((g'_{F'})^i)^*(P_{\eta'}) = P_{\eta'} \ ,$$

for any $i = 0, 1, \ldots, n-1$. Hence,

$$\Phi_{X'}(q) = \Phi_{X'}(\Delta_{X'} - p) =$$

$$= \Phi_{X'}(\Delta_{X'}) - \frac{1}{n}\sum_{i=0}^{n-1} \Phi_{X'}((\mathrm{id}_{X'} \times (g')^i)^*(\Delta_{X'})) =$$

$$= P_\eta - \frac{1}{n} \cdot n(P_\eta) = 0 \ .$$

Let now π_2^{tr} be the projector on X' determining $t^2(X')$. The kernel $\ker(\Phi_{X'})$ is generated by those algebraic cycles which are not dominant over the first factor, [11], and this kernel is a subgroup in the kernel of the homomorphism

$$CH^2(X' \times X') \longrightarrow \mathrm{End}(t^2(X'))$$

$$a \mapsto \pi_2^{\mathrm{tr}} \circ a \circ \pi_2^{\mathrm{tr}} \ ,$$

loc.cit. As $\Phi_{X'}(q) = 0$, we get $\pi_2^{\mathrm{tr}} \circ q \circ \pi_2^{\mathrm{tr}} = 0$, whence

$$\pi_2^{\mathrm{tr}} \circ p \circ \pi_2^{\mathrm{tr}} = \pi_2^{\mathrm{tr}} \ .$$

Since f' induces an isomorphism on the algebraic parts in the second Weil cohomology H^2, the projector p identifies the transcendental pieces in the Chow motives $M(Y')$ and $M(X')$, as needed. $\square$

Remark 3. Lemma 1 shows that if one of the three equivalent conditions in Lemma 2 holds true for the surface X over k, then all three equivalent conditions in Lemma 2 hold true also for the surface X_L, where L is an algebraically closed field extension of the field k.

Remark 4. Suppose the Bloch-Beilinson conjecture holds true over $\mathbb{C}$. Then (i), and so (ii), (iii) and (iv) in Lemma 2 hold true for X_L, for any algebraically closed sub-extension $k \subset L \subset \mathbb{C}$. Indeed, if the Bloch-Beilinson conjecture is true, then all Chow motives (with coefficients in $\mathbb{Q}$) over $\mathbb{C}$ are finite-dimensional, see [1]. Then all Chow motives over L are also finite-dimensional. As there

are no phantom sub-motives in finite-dimensional motives by Kimura's theorem, [12], the motive N_L vanishes in $\mathscr{CM}(L)$, so that (i) holds true in Lemma 2. In particular, if $n = 2$, i.e. $g_{\mathbb{C}}$ is a Nikulin involution on $X_{\mathbb{C}}$, the Bloch-Beilinson conjecture implies that the action of g_L on $CH^2(X_L)$ must be the identity.

4. Geometrical setup

Let k be an algebraically closed subfield in $\mathbb{C}$. All what will be going on in this section will be over the field k. Our purpose is to give a suitable geometrical description of $K3$-surfaces with Nikulin involutions corresponding to $\rho = 9$, $d = 3$ case, in terms of the classification in [7].

So, let S_0 be a $K3$-surface over k with a Nikulin involution

$$\tau : S_0 \longrightarrow S_0 \ ,$$

and such that

$$\rho = 9 \qquad \& \qquad d = 3 \ .$$

Then S_0 can be realized as a complete intersection,

$$S_0 = \mathscr{C}_0 \cap \mathscr{Q}_0 \ ,$$

of a cubic $\mathscr{C}_0$ and a quadric $\mathscr{Q}_0$ in $\mathbb{P}^4$, see [7, 3.3]. The involution τ can be extended to a regular involution

$$\tau : \mathbb{P}^4 \longrightarrow \mathbb{P}^4 \ ,$$

given in suitable coordinates

$$(x) = (x_0 : ... : x_4)$$

by the formula

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (-x_0 : -x_1 : x_2 : x_3 : x_4) \ , \tag{1}$$

and such that the quadric $\mathscr{Q}_0$ and the cubic $\mathscr{C}_0$ will be invariant under the global involution τ.

Vice versa, if $\mathscr{C}_0$ and $\mathscr{Q}_0$ are a smooth cubic and quadric in $\mathbb{P}^4$, both invariant under the involution τ, their intersection $S_0 = \mathscr{C}_0 \cap Q_0$ will be a $K3$-surface with algebraic Nikulin involution $\tau = \tau|_{S_0}$, loc.cit.

The fixed locus of the involution τ in the projective space $\mathbb{P}^4$ is a disjoint union of a line l_τ and a plane Π_τ in $\mathbb{P}^4$,

$$(\mathbb{P}^4)^\tau = l_\tau \sqcup \Pi_\tau \ ,$$

where

$$l_\tau : x_2 = x_3 = x_4 = 0 \ , \qquad \Pi_\tau : x_0 = x_1 = 0 \ . \tag{2}$$

Let now V be a vector space of dimension 5, such that

$$\mathbb{P}^4 = \mathbb{P}(V) \ .$$

The formula (1) shows that the involution τ lifts to an involution

$$\tau : V \longrightarrow V \ ,$$

which induces involutions

$$\tau_2 : \mathrm{Sym}^2 \check{V} \to \mathrm{Sym}^2 \check{V} \qquad \text{and} \qquad \tau_3 : \mathrm{Sym}^3 \check{V} \to \mathrm{Sym}^3 \check{V} \ ,$$

where Sym^n stays for the n-th symmetric power of a vector space over k, and

$$\check{V} = \mathrm{Hom}_k(V, k)$$

is a k-vector space dual to the space V. Let

$$(\mathrm{Sym}^2\check{V})_+ = \{F \in \mathrm{Sym}^2\check{V} \mid \tau_2(F) = F\}$$

and

$$(\mathrm{Sym}^3\check{V})_+ = \{\Phi \in \mathrm{Sym}^3\check{V} \mid \tau_3(\Phi) = \Phi\}$$

be the subspaces of forms, on which the involutions τ_2 and τ_3 act identically. From (1) it follows that these subspaces are given by the formulae

$$(\mathrm{Sym}^2\check{V})_+ = \{F \in \mathrm{Sym}^2\check{V} |\ F = \alpha_{00}x_0^2 + \alpha_{11}x_1^2 + \alpha_{01}x_0x_1 + f_2(x_2, x_3, x_4)\} \tag{3}$$

$$\begin{aligned}(\mathrm{Sym}^3\check{V})_+ = \{\Phi \in \mathrm{Sym}^3\check{V} \mid \Phi = l_{00}(x_2, x_3, x_4)x_0^2 + l_{11}(x_2, x_3, x_4)x_1^2 \\ +l_{01}(x_2, x_3, x_4)x_0x_1 + f_3(x_2, x_3, x_4)\} \ ,\end{aligned} \tag{4}$$

where α_{ij} are constants, l_{ij} are linear forms, f_2 and f_3 are homogeneous polynomials of degree 2 and 3 respectively. In [7, 3.3] it was shown that the cubic $\mathscr{C}_0$ and the quadric $\mathscr{Q}_0$ can be chosen to be defined by forms

$$\mathscr{C}_0 = \{\Phi_0 = 0\} \ , \qquad \Phi_0 \in (\mathrm{Sym}^3\check{V})_+ \ ,$$

$$Q_0 = \{F_0 = 0\} \ , \qquad F_0 \in \mathrm{Sym}^2\check{V}_+ \ .$$

For short, let

$$\mathscr{L}_2 = \mathbb{P}((\mathrm{Sym}^2\check{V})_+) \quad \text{and} \quad \mathscr{L}_3 = \mathbb{P}((\mathrm{Sym}^3\check{V})_+) \ .$$

Then,

$$\mathscr{Q}_0 \in \mathscr{L}_2 \subset |\mathscr{O}(2)| \quad \text{and} \quad \mathscr{C}_0 \in \mathscr{L}_3 \subset |\mathscr{O}(3)| \ ,$$

where

$$\mathscr{O} = \mathscr{O}_{\mathbb{P}^4}$$

is the structure sheaf on $\mathbb{P}^4$, and $\mathscr{F}(n)$ stands for the n-twist of a coherent sheaf $\mathscr{F}$ on a scheme.

Notice that the quadric $\mathscr{Q}_0$ is uniquely determined by the condition $S_0 \subset \mathscr{Q}_0$, i.e. the linear series

$$|\mathscr{I}_{S_0}(2)|$$

consists of the only quadric $\mathscr{Q}_0$, where $\mathscr{I}_{S_0}$ is the sheaf of ideals of the surface S_0 in $\mathbb{P}^4$. In contrast, the cubic $\mathscr{C}_0$ is not uniquely determined by the condition $S_0 \subset \mathscr{C}_0$. The linear series

$$\mathscr{L} = |\mathscr{I}_{S_0}(3)|$$

has dimension 5 and contains a linear subseries

$$\mathscr{L}_0 = \{\mathscr{C} \in \mathscr{L} \mid \mathscr{C} = \mathscr{Q}_0 \cup \mathbb{P}^3 \ , \ \mathbb{P}^3 \subset \mathbb{P}^4\} \simeq \check{\mathbb{P}}^4 \ ,$$

$$\mathscr{L}_0 = \{\mathscr{C} \in \mathscr{L} \mid \mathscr{C} = \mathscr{Q}_0 \cup H \ , \ \text{for a hyperplane } H \subset \mathbb{P}^4\} \simeq \check{\mathbb{P}}^4 \ ,$$

so that any cubic $\mathscr{C} \in \mathscr{L} \smallsetminus \mathscr{L}_0$ is irreducible and

$$S_0 = \mathscr{C} \cap \mathscr{Q}_0$$

for some $\mathscr{C}$ in $\mathscr{L} \smallsetminus \mathscr{L}_0$.

Since S_0 is smooth, any cubic $\mathscr{C} \in \mathscr{L} \smallsetminus \mathscr{L}_0$ is smooth along S_0. By Bertini's theorem, a general cubic in $\mathscr{L} \smallsetminus \mathscr{L}_0$ is smooth. We thus may well assume that $\mathscr{C}_0$ is smooth.

Consider a linear subseries

$$\mathscr{M}_3 = \{\mathscr{C} \in \mathscr{L}_3 \mid S_0 \subset \mathscr{C}\}$$

and its subseries

$$\mathscr{N}_3 = \{\mathscr{Q}_0 \cup H \in \mathscr{M}_3 \mid H \text{ a hyperplane containing } l_\tau\} \cong \mathbb{P}^2.$$

One can see then that

$$\mathscr{M}_3 = \operatorname{span}_{\mathscr{L}_3}(\mathscr{N}_3, \mathscr{C}_0) = \mathbb{P}(W)$$

for some vector space W of dimension 4 over k.

Lemma 5. *For any two points P and Q on S_0 there exists a cubic $\mathscr{C}$ in $\mathscr{L}_3 \smallsetminus \mathscr{M}_3$, such that $P, Q \in \mathscr{C} \cap \mathscr{Q}_0$.*

Proof. Look at a linear series Σ of curves on S_0 cut out cubics from $\mathscr{L}_3 \smallsetminus \mathscr{M}_3$,

$$\Sigma = \{C \subset S_0 \mid C = S_0 \cap \mathscr{C}\ ,\ \ \mathscr{C} \in \mathscr{L}_3 \smallsetminus \mathscr{M}_3\}\ .$$

As $\mathscr{L}_3 \cong \mathbb{P}^{18}$, it follows from the above that

$$\Sigma = \mathbb{P}((\operatorname{Sym}^3 \check{V})_+/W) \cong \mathbb{P}^{14}\ .$$

For any point P in S_0 we define a hyperplane

$$\Sigma_P = \{C \in \Sigma \mid x \in C\}$$

in the projective space Σ. As $\dim(\Sigma) = 14$, for any two distinct points P and Q in S_0 we have that

$$\dim(\Sigma_P \cap \Sigma_Q) \geq 12\ .$$

It gives that there exists a cubic $\mathscr{C}$ in $\mathscr{L}_3 \smallsetminus \mathscr{M}_3$ passing through P and Q. □

Lemma 6. *Any cubic $\mathscr{C}$ in $\mathscr{L}_3$ contains the line l_τ. Moreover, l_τ is actually the base locus of the linear series $\mathscr{L}_3$,*

$$l_\tau = \bigcap_{\mathscr{C} \in \mathscr{L}_3} \mathscr{C}\ . \tag{5}$$

Proof. That any cubic $\mathscr{C}$ in $\mathscr{L}_3$ contains the line l_τ is a consequence of (2) and (4). From (1) and (4) it follows that $\mathscr{L}_3$ is spanned by two subseries

$$\mathscr{L}_{3,i} = \mathbb{P}(V_i)\ , \quad i = 1, 2\ ,$$

where

$$V_1 = \{\Phi \in \operatorname{Sym}^3 \check{V} \mid \Phi = l_{00}(x_2, x_3, x_4)x_0^2 + l_{11}(x_2, x_3, x_4)x_1^2 + l_{01}(x_2, x_3, x_4)x_0x_1\}\ ,$$

$$V_2 = \{\Phi \in \operatorname{Sym}^3 \check{V} \mid \Phi = f_3(x_2, x_3, x_4)\}\ ,$$

and the forms l_{ij}, f_3 are those described above. A subgroup

$$G = \{g \in \mathrm{PGL}(5) \mid g(l_\tau) = l_\tau \quad \text{and} \quad g(\Pi_\tau) = \Pi_\tau\}$$

acts transitively on the set

$$\mathbb{P}^4 \smallsetminus (l_\tau \sqcup \Pi_\tau)\ .$$

It also acts naturally on the linear series $|\mathscr{O}(3)|$ and fixes the subspaces $\mathscr{L}_{3,1}$ and $\mathscr{L}_{3,2}$ in it. Hence, $\mathscr{L}_3$ is fixed under the G-action too.

Let

$$P_0 = (1:1:1:1:1)$$

be such a special point in $\mathbb{P}^4$. From (4) it follows that the subseries

$$\mathscr{L}_{3,\,P_0} = \{\mathscr{C} \in \mathscr{L}_3 \mid P_0 \in \mathscr{C}\}$$

is a proper hyperplane in $\mathscr{L}_3$. Since G acts transitively on $\mathbb{P}^4 \smallsetminus (l_\tau \sqcup \Pi_\tau)$, it follows that, for any other point

$$P \in \mathbb{P}^4 \smallsetminus (l_\tau \sqcup \Pi_\tau)\ ,$$

the subseries

$$\mathscr{L}_{3,\,P} = \{\mathscr{C} \in \mathscr{L}_3 \mid P \in \mathscr{C}\}$$

is also a proper hyperplane of $\mathscr{L}$, whence we get (5).

□

Now we need to show that a general quadric $\mathscr{Q}$ in the linear series

$$\mathscr{L}_2 = \mathbb{P}((\mathrm{Sym}^2\check{V})_+)$$

is smooth. Indeed, from (3) it follows that the series $\mathscr{L}_2$ is spanned by the two subseries

$$\mathscr{L}_{2,1} = \{\mathscr{Q} \in |\mathscr{O}(2)| \ \mid\ \mathrm{Sing}\,\mathscr{Q} \supset l_\tau\} \simeq \mathbb{P}(W_1)$$

and

$$\mathscr{L}_{2,2} = \{\mathscr{Q} \in |\mathscr{O}(2)| \ \mid\ \mathrm{Sing}\,\mathscr{Q} \supset \Pi_\tau\} \simeq \mathbb{P}(W_2)\ ,$$

where

$$W_1 = \{F \in \mathrm{Sym}^2\check{V} \mid F = f_2(x_2, x_3, x_4)\}\ ,$$

$$W_2 = \{F \in \mathrm{Sym}^2\check{V} \mid F = \alpha_{00}x_0^2 + \alpha_{11}x_1^2 + \alpha_{01}x_0x_1\}\ .$$

and α_{ij} and g_2 are as defined above.

In particular, a general quadric $\mathscr{Q}$ in $\mathscr{L}_{2,1}$ is a cone with a vertex being l_τ, and thus Q is smooth along $\mathscr{Q} \cap \Pi_\tau$. Respectively, a general quadric $\mathscr{Q}$ in $\mathscr{L}_{2,2}$ is a cone with a vertex Π_τ, and, therefore, $\mathscr{Q}$ is smooth at the two points of $\mathscr{Q} \cap l_\tau$. In view of Bertini's theorem, this description shows that a general quadric $\mathscr{Q}$ from $\mathscr{L}_2$ is smooth.

We can now take a general smooth cubic $\mathscr{C}$ in $\mathscr{L}_3$, a general smooth quadric $\mathscr{Q}$ in $\mathscr{L}_2$, and consider a surface

$$S = \mathscr{C} \cap \mathscr{Q}\ .$$

Naturally, specializing $\mathscr{C}$ to $\mathscr{C}_0$ and Q to Q_0, we specialize also S to the smooth surface S_0. Hence, for a general choice of $\mathscr{C}$ and $\mathscr{Q}$, the surface S is a smooth complete intersection of a τ-invariant cubic and a τ-invariant quadric in $\mathbb{P}^4$, so it is a τ-invariant $K3$-surface over the ground field k. Since, by construction, both S and S_0 belong to an irreducible family parametrized by an open subset of $\mathscr{L}_3 \times \mathscr{L}_2$, and τ is a Nikulin involution, it follows that the involution

$$\tau = \tau|_S : S \longrightarrow S$$

is also a Nikulin involution. Summarizing, one can prove the following lemma.

Lemma 7. *(i) For a general pencil of quadrics*

$$|\mathscr{Q}_t|_{t\in\mathbb{P}^1} \subset \mathscr{L}_2 ,$$

and for any two distinct smooth quadrics $\mathscr{Q}_0$ and $\mathscr{Q}_1$, where $0, 1 \in \mathbb{P}^1$, the surface

$$Y = \mathscr{Q}_0 \cap \mathscr{Q}_1$$

is smooth and not intersecting the line l_τ.

(ii) For a general smooth cubic $\mathscr{C} \in \mathscr{L}$, and for the above quadrics $\mathscr{Q}_0$ and $\mathscr{Q}_1$, the curve

$$Z = \mathscr{C} \cap Y$$

is smooth, and it does not intersect the line l_τ and the plane Π_τ. Hence, the induced involution

$$\tau : Z \longrightarrow Z$$

has no fixed points.

(iii) The restriction

$$|S_t|_{t\in\mathbb{P}^1}$$

of the pencil $|\mathscr{Q}_t|_{t\in\mathbb{P}^1}$ on $\mathscr{C}$ is a pencil of $K3$-surfaces on $\mathscr{C}$ endowed with fibre-wise Nikulin involutions

$$\tau : S_t \to S_t , \quad t \in \mathbb{P}^1 .$$

Proof. (i) Let

$$\mathscr{Q}_{00} : x_0^2 + g_2(x_2, x_3, x_4) = 0 \qquad \text{and} \qquad \mathscr{Q}_{10} : x_1^2 + h_2(x_2, x_3, x_4) = 0$$

be two quadrics in $\mathscr{L}_2$. A straightforward computation shows that the surface $Y_0 = \mathscr{Q}_{00} \cap \mathscr{Q}_{10}$ is smooth and does not intersect the line l_τ, if the conics in Π_τ with the equations $g_2(x_2, x_3, x_4) = 0$ and $h_2(x_2, x_3, x_4) = 0$ are smooth and intersect transversally at 4 points. Then, for general smooth quadrics $\mathscr{Q}_0$ and $\mathscr{Q}_1$ in the series $\mathscr{L}_2$ the surface $Y = \mathscr{Q}_0 \cap \mathscr{Q}_1$ is smooth and $Y \cap l_\tau = \emptyset$.

(ii) Consider the restriction

$$\mathscr{L}_3|_Y = \{\mathscr{C} \cap Y\}_{\mathscr{C}\in\mathscr{L}_3}$$

of the linear series $\mathscr{L}_3$ on the surface Y. Since $Y \cap l_\tau = \emptyset$, from (5) it then follows that $\mathscr{L}_3|_Y$ has no base points. Hence, by Bertini's theorem, a general member Z of this series is smooth and does not intersect l_τ.

(iii) This is clear since each smooth $K3$-surface $S = \mathscr{C} \cap \mathscr{Q}$ is endowed with a Nikulin involution induced by the involution τ on $\mathbb{P}^4$. □

Now let us consider the projection of $\mathbb{P}^4$ onto Π_τ from the line l_τ, and restrict it onto a general smooth cubic $\mathscr{C}$ in the series L,

$$p : \mathscr{C} \dashrightarrow \mathbb{P}^2 ,$$

where $\mathbb{P}^2$ can be naturally identified with the plane Π_τ. Blowing up $\mathscr{C}$ at the indeterminacy locus l_τ, we obtain a regular morphism $\hat{p}$ included in to the commutative diagram

$$\begin{tikzcd} \hat{\mathscr{C}} \arrow[r] \arrow[dr, "\hat{p}"'] & \mathscr{C} \arrow[d, dashed, "p"] \\ & \Pi_\tau \end{tikzcd}$$

Let

$$C = \{P \in \mathbb{P}^2 \mid \hat{p}^{-1}(P) \text{ is a pair of crossing lines on } \hat{\mathscr{C}}\}\ .$$

be the discriminant curve of the projection $\hat{p}$. This is an algebraic curve of degree 5 in Π_τ.

Let now

$$\mathscr{F} = \{l \in \mathrm{Gr}(2,5) \,|\, l \subset \mathscr{C}\}$$

be the Fano surface of lines on $\mathscr{C}$, and, following [15], let's define the sets

$$\mathscr{F}_0 = \{l \in \mathscr{F} \,|\, \exists \text{ a plane } \Pi \text{ in } \mathbb{P}^4, \text{ such that } \mathscr{C} \cdot \Pi = 2l + l'\}$$

and

$$\mathscr{F}'_0 = \{l \in \mathscr{F} \,|\, \exists \text{ a plane } \Pi \text{ in } \mathbb{P}^4, \text{ such that } \mathscr{C} \cdot \Pi = l + 2l'\}$$

Then, in terms of the analysis of lines on $\mathscr{C}$ given in loc.cit., the line l_τ does not sit in the set $\mathscr{F}_0$, but is does belong to the set $\mathscr{F}'_0$.

An elementary computation shows that the discriminant curve C consists of 2 irreducible components,

$$C = C_2 \cup C_3\ ,$$

where C_2 is a conic defined by the equation

$$C_2 : 4l_{00}(x_2,x_3,x_4)l_{11}(x_2,x_3,x_4) - l_{01}(x_2,x_3,x_4)^2 = 0\ ,$$

and C_3 is a cubic defined by the equation

$$f_3(x_2,x_3,x_4) = 0$$

in Π_τ, where l_{00}, l_{11}, l_{01} and f_3 are taken from (4). For the general choice of l_{00}, l_{11}, l_{01} and f_3 the curves C_2 and C_3 are smooth and intersect each other transversally at 6 distinct points in Π_τ. Looking at the equation $4l_{00}l_{11} - l_{01}^2 = 0$ for C_2 as an equation in $\mathbb{P}^4$, it defines a cone

$$\mathscr{K} = \mathrm{cone}(l_\tau, C_2)$$

with the vertex l_τ over the conic C_2. Since C_2 is smooth, the singularities of the quadric cone $\mathscr{K}$ are concentrated in the line l_τ. This $\mathscr{K}$ is τ-invariant as the line l_τ, as well as the conic C_2, both consist of fixed points of the involution τ.

Lemma 8. *(i) For a general smooth quadric $\mathscr{Q}$ from $\mathscr{L}_2$ the intersection of $\mathscr{Q}$ with the line l_τ consists of 2 distinct points P_1 and P_2. The surface*

$$Y = \mathscr{K} \cap \mathscr{Q}$$

is singular, it intersects l_τ at two points P_1 and P_2, and

$$\mathrm{Sing}(Y) = \{P_1, P_2\}\ .$$

(ii) Let

$$Y = \mathscr{K} \cap \mathscr{Q} \qquad \text{and} \qquad Z = \mathscr{C} \cap Y \ .$$

Then there is a finite collection of ordinary double points on the curve Z.

(iii) If now

$$|S_t|_{t \in \mathbb{P}^1}$$

is a restriction of the a pencil of quadrics through the singular quadric K and the general smooth quadric $\mathscr{Q}$ from $\mathscr{L}_2$, then $|S_t|_{t \in \mathbb{P}^1}$ is again a pencil of $K3$-surfaces on $\mathscr{C}$ endowed with Nikulin involutions

$$\tau : S_t \longrightarrow S_t \ , \ \ t \in \mathbb{P}^1 \ .$$

Proof. The proof is similar to the proof of Lemma 7. □

5. Pencils of surfaces on threefolds with small A^2

The previous geometrical setup can be generalized. In this section we will be working over $\mathbb{C}$. Let $\mathscr{X}$ be a smooth projective threefold over $\mathbb{C}$ and a rational map

$$f : \mathscr{X} \dashrightarrow \mathbb{P}^1$$

interpreted as a pencil of surfaces on $\mathscr{X}$. Let B be a base locus of the pencil f, and suppose that B is reduced, irreducible and smooth. Let

$$\begin{tikzcd} \tilde{\mathscr{X}} \arrow[r, "s"] \arrow[dr, "\tilde f"'] & \mathscr{X} \arrow[d, dashed, "f"] \\ & \mathbb{P}^1 \end{tikzcd}$$

be a blow up of $\mathscr{X}$ at B. For any $t \in \mathbb{P}^1$ let X_t be the geometric fibre of the pencil $\tilde f$ over the point t, and let

$$i_t : X_t \hookrightarrow \tilde{\mathscr{X}}$$

be a corresponding closed imbedding. The surface X_t is not necessarily smooth, so that we need also a desingularization

$$X_t' \longrightarrow X_t \ ,$$

which is by convention an identity if X_t is smooth, and let

$$i_t' : X_t' \hookrightarrow \tilde{\mathscr{X}}$$

be a composition of i_t with the desingularization.

Consider the blow up Cartesian square in the category of schemes over $\mathbb{C}$:

$$\begin{tikzcd} \tilde B \arrow[r, "\tilde i"] \arrow[d, "r"'] & \tilde{\mathscr{X}} \arrow[d, "s"] \\ B \arrow[r, "i"] & \mathscr{X} \end{tikzcd}$$

Here $\tilde{B}$ is a proper transform of the curve B under the blow up s, $i : B \hookrightarrow \mathscr{X}$ and $i : \tilde{B} \hookrightarrow \tilde{\mathscr{X}}$ are closed embeddings. By Proposition 6.7 in [6], we have a short exact sequence of abelian groups

$$0 \to CH_1(B) \xrightarrow{a} CH_1(\tilde{B}) \oplus CH^2(\mathscr{X}) \xrightarrow{b} CH^2(\tilde{\mathscr{X}}) \to 0 ,$$

where for any irreducible variety V of dimension d we write $CH_n(V) = CH^{d-n}(V)$. The group $CH_1(B) = CH^0(B)$ is obviously isomorphic to $\mathbb{Q}$. Restricting this short exact sequence on cycle classes algebraically equivalent to 0 we obtain an isomorphism

$$A^2(\tilde{\mathscr{X}}) \simeq A^2(\mathscr{X}) \oplus A^1(B) ,$$

which describes the link between algebraic cycles on the original threefold $\mathscr{X}$ and its blow up $\tilde{\mathscr{X}}$.

We need now to impose two basic assumptions on our threefold $\mathscr{X}$. The first one says that the group $A^2_{\mathbb{Z}}(\mathscr{X})$ is (weakly) representable by the intermediate Jacobian:

Assumption A: The Abel-Jacobi homomorphism

$$AJ : A^2_{\mathbb{Z}}(\mathscr{X}) \longrightarrow J^3_{\mathrm{alg}}(\mathscr{X})$$

is an isomorphism.

To state the second assumption we need to enrich the picture by endowing $\mathscr{X}$ with an action of a finite cyclic group G which acts fibre-wise, i.e. all the fibres X_t of the above pencil are G-equivariant. Assumption A allows to introduce two subgroups $A^2_{\mathbb{Z}}(\mathscr{X})^G$ and $A^2_{\mathbb{Z}}(\mathscr{X})^\sharp$ which correspond to, respectively, $J^3_{\mathrm{alg}}(\mathscr{X})^G$ and $J^3_{\mathrm{alg}}(\mathscr{X})^\sharp$ under the above Abel-Jacobi isomorphism.

Since now let the sign $\natural$ stay for G *and $\sharp$ simultaneously*

Assumption B:

$$A^2(\mathscr{X})^\natural = 0$$

(still the group $A^2_{\mathbb{Z}}(\mathscr{X})^\natural$ may well have torsion).

Taking into account Assumptions A and B, we see that

$$A^1(B)^\natural \simeq A^2(\tilde{\mathscr{X}})^\natural ,$$

and this isomorphism nothing but the $\natural$-version of a composition of the pull-back $A^1(B) \to A^1(\tilde{B}) = A_1(\tilde{B})$ and the push-forward $A_1(\tilde{B}) \to A_1(\tilde{\mathscr{X}}) = A^2(\tilde{\mathscr{X}})$. The group $A^1_{\mathbb{Z}}(B)$ is isomorphic to the Jacobian variety of the curve B,

$$A^1_{\mathbb{Z}}(B) \simeq J(B) .$$

Then we also have an isomorphism

$$J(B)^\natural_{\mathbb{Q}} \simeq J^3_{\mathrm{alg}}(\mathscr{X})^\natural_{\mathbb{Q}} .$$

Fix a prime l. For any field F and any smooth projective variety V over F let $H^p_{\acute{e}t}(V_{\bar{F}}, \mathbb{Q}_l(q))$ be the p-th étale cohomology group of V twisted by q, where

$\bar{F}$ is the algebraic closure of the field F. It is important to mention that l-adic étale cohomology groups form a Weil cohomology theory over F.

Let η be the generic point of the projective line $\mathbb{P}^1$. Since the whole pencil is G-equivariant, the group G acts also on the scheme-theoretical generic fibre X_η of the fibration $\tilde{f} : \tilde{\mathscr{X}} \to \mathbb{P}^1$. For any $t \in \mathbb{P}^1$, such that the fibre X_t is smooth, there is a commutative diagram

$$\begin{tikzcd} CH^1(X_{\bar{\eta}}) \arrow[r, "cl"] \arrow[d, "\mathrm{sp}"'] & H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1)) \arrow[d, "\mathrm{sp}"] \\ CH^1(X_t) \arrow[r, "cl"] & H^2_{\acute{e}t}(X_t, \mathbb{Q}_l(1)) \end{tikzcd}$$

where the vertical homomorphisms are specializations and the horizontal ones are cycle class maps into étale cohomology groups, [6]. Moreover, it is well known that the right vertical homomorphism is an isomorphism. It follows that if the group $H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1))$ is algebraic, i.e. the top horizontal cycle class map is onto, then so is the group $H^2_{\acute{e}t}(X_t, \mathbb{Q}_l(1))$. The group $H^2_{\acute{e}t}(X_t, \mathbb{Q}_l(1))$ is isomorphic to the Betti cohomology group $H^2(X_t, \mathbb{Q})$. Therefore, if $H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1))$ is algebraic is equivalent to say that $H^2(X_t, \mathbb{Q})$ is algebraic. This last thing is, in turn, equivalent to say that $H^{2,0}(X_t) = 0$ for all points t in the Zariski open subset U in $\mathbb{P}^1$, such that X_t is smooth for all point s $t \in U$. In fact, one can also show the converse statement – if there exists a Zariski open $U \subset \mathbb{P}^1$ such that X_t is smooth and $H^{2,0}(X_t) = 0$ for all $t \in U$, then $H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1))$ is algebraic.

If we take into account the action of a group G in the fibres of the regular map $\tilde{f} : \tilde{\mathscr{X}} \to \mathbb{P}^1$, we obtain the following picture. First of all, we have the group of invariants $H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1))^G$. Secondly, there is a $\cup$ -product in $H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1))$, which gives rise to the orthogonal complement $H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1))^\sharp = (H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1))^G)^\perp$. Then the previous equivalences can be refined by saying that $H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1))^\natural$ is algebraic if and only if $\exists$ a Zariski open $U \subset \mathbb{P}^1$, such that $H^2(X_t, \mathbb{Q})^\natural$ is algebraic for any $t \in U$.

Finally, we need two more assumptions to be imposed onto the above geometrical input.

Assumption C. The group $H^2_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l(1))^\natural$ is algebraic.

Embedding the function field over $\mathbb{P}^1$ into $\mathbb{C}$, this third Assumption C is a restriction imposed on the behaviour of holomorphic 2-forms of the generic fibre.

Assumption D. The generic fibre X_η is regular, i.e.

$$H^1_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Q}_l) = 0 ,$$

in terms of the first étale cohomology group of the generic fibre.

We believe that Assumption D isn't important and can be eliminated at the cost of extra technicalities around the Picard and Albanese motives of the generic fibre.

6. Voisin's theorem

Keeping notation of the last section, our aim is now to generalize Voisin's approach originally given in [20] and then described also in [21].

For any $t \in \mathbb{P}^1$ let $j_t : B \hookrightarrow X_t$ be the corresponding closed embedding, and let

$$j'_t : B \hookrightarrow X'_t$$

be the embedding of the base locus B into the desingularization of the surface X_t. Certainly, if X_t is smooth then $X'_t = X_t$ and $j'_t = j_t$.

Theorem 9. *(Prop. 11.15 in* [21]*) Under the above Assumptions A, B, C and D, the group $A^1(B)^\natural$ is generated by the cycle classes of type $(j'_t)^*(\alpha)$, for $\alpha \in A^1(X'_t)^\natural$.*

Proof. Let U be a Zariski open subset in $\mathbb{P}^1$, such that the fibres over points in U are smooth, and let

$$\tilde{f}_U : \tilde{\mathscr{X}}_U \longrightarrow U$$

be the corresponding pull-back on U. Then there exists the Picard scheme

$$P_U = \mathrm{Pic}(\tilde{\mathscr{X}}_U/U)$$

of the relative scheme $\tilde{\mathscr{X}}_U/U$, coming together with an étale morphism

$$P_U \longrightarrow U\ ,$$

see [5]. For any $t \in U$ the fibre $(P_U)_t$ of the morphism $P_U \to U$ is nothing but the Picard scheme $\mathrm{Pic}(X_t)$ of the fibre X_t, so naturally isomorphic to the group $CH^1_{\mathbb{Z}}(X_t)$. The group G acts on the scheme P_U fibre-wise, so that we obtain the corresponding G-invariant subschemes P_U^G and $(P_U)_t^G$, for each $t \in U$, such that the fibre of the corresponding regular morphism

$$P_U^G \longrightarrow U$$

is $(P_U)_t^G$. Since G acts fibre-wise on $\tilde{\mathscr{X}}$ and $(P_U)_t$ is isomorphic to $CH^1_{\mathbb{Z}}(X_t)$, the fibre $(P_U)_t^G$ is naturally isomorphic to $CH^1_{\mathbb{Z}}(X_t)^G$.

Let $P_U^G \hookrightarrow P_U$ be the obvious closed embedding over U, and let $P_U^\vee \to (P_U^G)^\vee$ be the dual morphism, see [14]. The relative abelian scheme $P_U \to U$ admits a relatively ample invertible sheaf $\mathscr{L}$, which determines a regular U-homomorphism $P_U \to P_U^\vee$ over U, loc.cit. Let $P_U^\sharp$ be the kernel of the composition $P_U \to P_U^\vee \to (P_U^G)^\vee$. Then we have a fibre-wise isogeny

$$P_U^G \times_U P_U^\sharp \longrightarrow P_U$$

over U (the reader is advised to apply the same sort of argument like on page 42 of [13]). Certainly, the fibres $(P_U^\sharp)_t$ naturally give rise to subgroups $CH^1_{\mathbb{Z}}(X_t)^\sharp$ in $CH^1(X_t)^\sharp$.

Since the scheme-theoretical generic fibre X_η is regular, so is the fibre X_t for each $t \in U$. Therefore, the connected component of zero, $\mathrm{Pic}^0(X_t)$, in the Picard scheme $\mathrm{Pic}(X_t)$, is trivial, i.e. $A^1_{\mathbb{Z}}(X_t) \simeq \mathrm{Pic}^0(X_t) = 0$, so that each scheme $(P_U)_t$ is isomorphic, as an abelian group, to the Néron-Severi group $NS(X_t)$, which is known to be finitely-generated. As homological equivalence coincides with the algebraic one in codimension 1, [6], it also means that each $(P_U)_t = NS(X_t)$ is a subgroup in $H^2(X_t, \mathbb{Z})$. Then, of course, $(P_U)_t^G \simeq NS(X_t)^G$, where $NS(X_t)^G$ is the intersection of $H^2(X_t, \mathbb{Z})^G$ with $NS(X_t)$ inside $H^2(X_t, \mathbb{Z})$,

and, similarly, $(P_U)^\sharp_t \simeq NS(X_t)^\sharp$, where $NS(X_t)^\sharp$ is the intersection of $H^2(X_t,\mathbb{Z})^\sharp$ with $NS(X_t)$ inside $H^2(X_t,\mathbb{Z})$. But as the group $H^2(X_t,\mathbb{Q})^G$ (respectively, the group $H^2(X_t,\mathbb{Q})^\sharp$) is algebraic in case (G) (respectively, in case ($\sharp$)), we obtain that

$$(P_U)^\sharp_t \simeq NS(X_t)^\natural = H^2(X_t,\mathbb{Z})^\natural \ .$$

As P_U is étale over U and $(P_U)_t$ is a discrete abelian group for each $t\in U$, the scheme P_U can be only a disjoint union $\mathscr{U}$ of étale curves over U. Since the group $H^2(X_t,\mathbb{Z})^\natural$ is finitely-generated, we can pick up a finite collection of smooth curves from $\mathscr{U}$, such that, if $W^\natural$ is a union of them, $W^\natural$ is a smooth and not necessarily connected curve, étale over U with respect to the restriction

$$g^\natural : W^\natural \longrightarrow U$$

of the above map $P_U\to U$ on $W^\natural$, having the property that the fibre $(g^\natural)^{-1}(t)$ generates the group $H^2(X_t,\mathbb{Z})^\natural$. Using appropriate compactifications of the curves $W^\natural$, we can also work with morphisms

$$\bar g^\natural : \bar W^\natural \longrightarrow \mathbb{P}^1 \ .$$

where the curves $\bar W^\natural$ are now not only smooth but also projective, and still not necessarily connected, étale curves over the projective line $\mathbb{P}^1$. Then $g^\natural$ can be also considered as a pull-back of $\bar g^\natural$ under the embedding of U into $\mathbb{P}^1$.

Let now $\mathbb{Z}$ be the constant sheaf in complex topology on $\bar W^\natural$. As $\mathbb{P}^1$ is locally contractible, in complex topology, the stalk of the direct image $R^0\bar g^\natural_*\mathbb{Z}$ at $t\in\mathbb{P}^1$ is isomorphic to the group $H^0(\bar W^\natural_t,\mathbb{Z})$, where $\bar W^\natural_t$ is fibre of the morphism $\bar g^\natural$ over $t\in\mathbb{P}^1$. If now $\mathbb{Z}$ is a constant sheaf in complex topology on $\tilde{\mathscr{X}}$ then, for the same reason, the stalk of the direct image $R^2\tilde f_*\mathbb{Z}$ at $t\in\mathbb{P}^1$ is isomorphic to the group $H^2(X'_t,\mathbb{Z})$.

Suppose $t\in U$. Then $X'_t = X_4$ and $\bar W^\natural_t = W^\natural_t$ is just a finite collection of points generating $H^2(X_t,\mathbb{Z})^\natural$. Respectively, $H^0(\bar W^\natural_t,\mathbb{Z})$ is a direct sum of a finite collection of copies of the group $\mathbb{Z}$. Moreover, $R^0\bar g^\natural_*\mathbb{Z}|_U = R^0g^\natural_*\mathbb{Z}$ and, if

$$f_U : \tilde{\mathscr{X}}_U \longrightarrow U$$

is a pull-back of the morphism $\tilde f$ with respect to the open embedding of U into $\mathbb{P}^1$, then $R^2\tilde f_*\mathbb{Z}|_U = R^2(\tilde f_U)_*\mathbb{Z}$.

The morphisms $g^\natural$ and f_U being étale induce locally trivial continuous mappings in complex topology. This is why the direct images $R^0g^\natural_*\mathbb{Z}$ and $R^2(\tilde f_U)_*\mathbb{Z}$ are local systems. Trivializing the maps $g^\natural$ and f_U by an appropriate covering $U=\cup U_i$, we define the sheaves $(R^2(\tilde f_U)_*\mathbb{Z}|_{U_i})^\natural$, which then can be pasted all together into sheaves $(R^2(\tilde f_U)_*\mathbb{Z})^\natural$ on U. As $W^\natural_t$, for each $t\in U$, generates $H^2(X_t,\mathbb{Z})^\natural$, using the same covering $U=\cup U_i$ we can also define a surjective map $R^0g^\natural_*\mathbb{Z}|_{U_i}\to (R^2(\tilde f_U)_*\mathbb{Z}|_{U_i})^\natural$, which then paste into a surjective morphism

$$\alpha : R^0g^\natural_*\mathbb{Z} \longrightarrow (R^2(\tilde f_U)_*\mathbb{Z})^\natural$$

of sheaves on the topological space U. It induces a surjection

$$\alpha_* : H^1(U,R^0g^\natural_*\mathbb{Z}) \to H^1(U,(R^2(\tilde f_U)_*\mathbb{Z})^\natural) \ .$$

One can show that, in our case, $R^1g^\natural\mathbb{Z}=0$ and $R^3(\tilde f_U)_*\mathbb{Z}=0$. As the corresponding Leray spectral sequences converge, we obtain two isomorphisms

$$H^1(U,R^0g^\natural_*\mathbb{Z}) \simeq H^1(W^\natural,\mathbb{Z}) \quad\text{and}\quad H^1(U,(R^2(\tilde f_U)_*\mathbb{Z})^\natural) \simeq H^3(\tilde f^{-1}(U),\mathbb{Z})^\natural \ .$$

Composing and precomposing the above surjection $H^1(U,\alpha)$ by means of these two isomorphisms, we obtain a new epimorphism

$$\alpha'_* : H^1(W^\natural, \mathbb{Z}) \longrightarrow H^3(\tilde{f}^{-1}(U), \mathbb{Z})^\natural \ .$$

Let

(6)
$$\begin{tikzcd} W^\natural \times_U \tilde{\mathscr{X}}_U \arrow[r, "p_2"] \arrow[d, "p_1"] & \tilde{\mathscr{X}}_U \arrow[d, "\tilde{f}_U"] \\ W^\natural \arrow[r, "g^\natural"] & U \end{tikzcd}$$

be the fibred product of $W^\natural$ and $\tilde{\mathscr{X}}_U$ over U. The morphism p_1 is flat as a base-change of the flat morphism $\tilde{f}_U$.

From the general machinery of Picard schemes over a base it follows that there exists a divisor $\mathscr{D}$ on $W^\natural \times_U \tilde{\mathscr{X}}_U$, such that, if P is a point on $W^\natural$ and $t = g^\natural(P)$ is the image of P on U, the intersection of $\mathscr{D}$ with the flat pull-back $p_1^*(P)$ is nothing but a divisor, whose class in $H^2(X_t, \mathbb{Z})^\natural$ corresponds to the point P under the above parametrization. It also follows that such a divisorial class is nothing but the value of the map α at P in the fibre over t. Since $W^\natural \times_U \tilde{\mathscr{X}}_U$ is a closed subscheme in $W^\natural \times \tilde{\mathscr{X}}_U$, the cycle $\mathscr{D}$ is a codimension 2 algebraic cycle in $W^\natural \times \tilde{\mathscr{X}}_U$,

$$\mathscr{D} \subset W^\natural \times \tilde{\mathscr{X}}_U \ .$$

Let also

$$\bar{\mathscr{D}} \subset \bar{W}^\natural \times \tilde{\mathscr{X}}$$

be the closure of $\mathscr{D}$ in $\bar{W}^\natural \times \tilde{\mathscr{X}}$, and let

$$\bar{\beta} = cl(\bar{\mathscr{D}}) \in H^4(\bar{W}^\natural \times \tilde{\mathscr{X}}, \mathbb{Q})$$

be the cohomology class of the cycle $\bar{\mathscr{D}}$.

As the Künneth formula holds true with coefficients in $\mathbb{Z}$ after killing torsion, see Theorem 11.38 on page 285 of the first volume of [21], we can also consider the $(1,3)$-Künneth component

$$\bar{\beta}(1,3) \in H^1(\bar{W}^\natural, \mathbb{Z}) \otimes H^3(\tilde{\mathscr{X}}, \mathbb{Z})$$

be the $(1,3)$-Künneth component of the class $\bar{\beta}$ in H^4. Let then

$$\bar{\beta}(1,3)_* : H^1(\bar{W}^\natural, \mathbb{Z}) \to H^3(\tilde{\mathscr{X}}, \mathbb{Z})$$

be a homomorphism of Hodge structures induced by the Künneth component $\bar{\beta}(1,3)$.

Recall that the curve $\bar{W}^\natural$ is smooth and projective, but not necessarily connected. Let then

$$\bar{W}^\natural = \bar{W}_1^\natural \cup \cdots \cup \bar{W}_n^\natural$$

be the connected components of $\bar{W}^\natural$, and let

$$\bar{\beta}(1,3)_{i,*} : H^1(\bar{W}_i^\natural, \mathbb{Z}) \to H^3(\tilde{\mathscr{X}}, \mathbb{Z})$$

be the restriction of $\bar{\beta}(1,3)_*$ on the i-th component of the group

$$H^1(\bar{W}^\natural, \mathbb{Z}) \simeq \oplus_i H^1(\bar{W}_i^\natural, \mathbb{Z}) \ .$$

Then, of course, $\bar{\beta}(1,3)_{i,*}$ is again a morphism of Hodge structures.

Since each $\bar{W}_i^{\natural}$ is a smooth projective curve over $\mathbb{C}$, we have the Jacobian $J^1(\bar{W}_i^{\natural})$ of it. Let

$$J^1(\bar{W}^{\natural}) = \oplus_{i=1}^n J^1(\bar{W}_i^{\natural}) \ .$$

The morphism $\bar{\beta}(1,3)_{i,*}$, being a morphism of Hodge structures, induces a holomorphic mapping

$$\bar{\beta}(1,3)_{i,*} : J^1(\bar{W}_i^{\natural}) \longrightarrow J^3(\tilde{\mathscr{X}}) \ .$$

By Theorem 12.17 on page 300 in the first volume of [21], applied in our disconnected case component-wise, we have that the last holomorphic mapping can be computed geometrically via the Abel-Jacobi map for the threefold $\tilde{\mathscr{X}}$.

The precise description of holomorphic map $\bar{\beta}(1,3)_{i,*}$ is as follows. Fix a point P_0 on the smooth projective curve $\bar{W}_i^{\natural}$. For any other point P on $\bar{W}_i^{\natural}$ let $[P-P_0]$ be the class of the degree zero cycle $P-P_0$ in the Jacobian. Let $t = \bar{g}^{\natural}(P)$ and $t_0 = \bar{g}^{\natural}(P_0)$, and let D_P and D_{P_0} be the divisors on the surfaces X'_t and X'_{t_0}, whose cohomology classes correspond to the points P and P_0 respectively. Let also

$$E_P = (i'_t)_*(D_P) \qquad \text{and} \qquad E_{P_0} = (i'_{t_0})_*(D_{P_0})$$

be the corresponding divisors in the threefold $\tilde{\mathscr{X}}$ obtained by push-forwarding them via the morphisms i'_t and i'_{t_0}, respectively. If now $q_1 : \bar{W}^{\natural} \times \tilde{\mathscr{X}} \to \bar{W}^{\natural}$ and $q_2 : \bar{W}^{\natural} \times \tilde{\mathscr{X}} \to \tilde{\mathscr{X}}$ are the projections, from the definition of $\mathscr{D}$ it then follows that E_P and E_{P_0} can be also computed as

$$E_P = (q_2)_* q_1^*(P) \qquad \text{and} \qquad E_{P_0} = (q_2)_* q_1^*(P_0) \ .$$

Therefore, E_P is homologous to E_{P_0} on the threefold $\tilde{\mathscr{X}}$, and then

$$\bar{\beta}(1,3)_{i,*}[P-P_0] = AJ(E_P - E_{P_0})$$

on the intermediate Jacobian $J^3(\tilde{\mathscr{X}})$, see Theorem 12.4 on page 294 of the first volume of [21].

One has isomorphisms

$$H^0(U, R^2(g^{\natural}p_1)_*\mathbb{Z}) \simeq \mathrm{Hom}(R^0 g^{\natural}_*\mathbb{Z}, R^2(\tilde{f}_U)_*\mathbb{Z})$$

and

$$H^0(U, R^2(g^{\natural}p_1)_*\mathbb{Z}) \simeq H^2(W^{\natural} \times_U \tilde{\mathscr{X}}_U, \mathbb{Z}) \ .$$

Composing we get an isomorphism

$$w : H^2(W^{\natural} \times_U \tilde{\mathscr{X}}_U, \mathbb{Z}) \stackrel{\sim}{\longrightarrow} \mathrm{Hom}(R^0 g^{\natural}_*\mathbb{Z}, R^2(\tilde{f}_U)_*\mathbb{Z})$$

The algebraic cycle $\bar{\mathscr{D}}$ is a divisor on $\bar{W}^{\natural} \times_{\mathbb{P}^1} \tilde{\mathscr{X}}$. Let

$$\bar{\gamma} = cl(\bar{\mathscr{D}}) \in H^2(\bar{W}^{\natural} \times_{\mathbb{P}^1} \tilde{\mathscr{X}}, \mathbb{Z})$$

be the its cohomology class, and let

$$\gamma \in H^2(W^{\natural} \times_U \tilde{\mathscr{X}}_U, \mathbb{Z})$$

be its restriction on Zariski open subset $W^{\natural} \times_U \tilde{\mathscr{X}}_U$. Then γ can be also considered as the cohomology class of the divisor $\mathscr{D}$ on $W^{\natural} \times_U \tilde{\mathscr{X}}_U$. The point here is that

$$w(\gamma) = \alpha \ .$$

It gives immediately that the diagram

$$
\begin{tikzcd}
H^1(\bar{W}^\natural, \mathbb{Z}) \arrow[r, "\bar{\beta}(1,3)_*"] \arrow[d, "r_1"] & H^3(\tilde{\mathscr{X}}, \mathbb{Z})^\natural \arrow[d, "r_2"] \\
H^1(W^\natural, \mathbb{Z}) \arrow[r, "\alpha'_*"] & H^3(\tilde{f}^{-1}(U), \mathbb{Z})^\natural
\end{tikzcd}
\tag{7}
$$

is commutative, where r_1 and r_2 are the restriction homomorphisms on cohomology groups. Tensoring this commutative square by $\mathbb{Q}$ we may also obtain its $\mathbb{Q}$-local version, which is better to work with from the point of view of the Hodge theory.

Due to Deligne's results, [4], the cohomology groups at the bottom of the $\mathbb{Q}$-localized commutative diagram (7) possess a mixed Hodge structure, such that their 0-weights are exactly the images of the vertical homomorphisms of the diagram, i.e.

$$W_0 H^1(W^\natural, \mathbb{Q}) = \mathrm{im}(r_1)$$

and

$$W_0 H^3(\tilde{f}^{-1}(U), \mathbb{Q})^\natural = \mathrm{im}(r_2)\ ,$$

see [21, Volume II, p. 321]. The power of mixed Hodge structures is such that morphisms between them are strict with respect to both Hodge and weight filtrations. That was proved by Deligne in [4]. Since α'_* is a morphism of mixed Hodge structures, we obtain, in particular, that

$$\alpha'_*(W_0 H^1(W^\natural, \mathbb{Q})) = \mathrm{im}(\alpha'_*) \cap W_0 H^3(\tilde{f}^{-1}(U), \mathbb{Q})^\natural\ .$$

Moreover, as α'_* is surjective, we obtain

$$\alpha'_*(W_0 H^1(W^\natural, \mathbb{Q})) = W_0 H^3(\tilde{f}^{-1}(U), \mathbb{Q})^\natural\ .$$

Therefore,

$$\mathrm{im}(r_2) = \mathrm{im}(r_2 \circ \bar{\beta}(1,3)_*)\ .$$

It follows then that the whole cohomology group $H^3(\tilde{\mathscr{X}}, \mathbb{Z})^\natural$ is generated by the kernel of the homomorphism r_2 and the image of the homomorphism $\beta(1,3)_*$. From a suitable localization sequence for cohomology groups, we know that $\ker(r_2)$ is generated by the images of the homomorphisms

$$(i'_t)_* : H^1(X'_t, \mathbb{Q})^\natural \to H^3(\tilde{\mathscr{X}}, \mathbb{Q})^\natural\ ,$$

where $t \in \mathbb{P}^1 \smallsetminus U$.

All these things together gives that the homomorphism

$$\pi : H^1(\bar{W}^\natural, \mathbb{Q}) \oplus (\oplus_{t \in \mathbb{P}^1 \smallsetminus U} H^1(X'_t, \mathbb{Q})^\natural) \to H^3(\tilde{\mathscr{X}}, \mathbb{Z})^\natural\ ,$$

induced by the above homomorphisms $\bar{\beta}(1,3)_*$ and $(i'_t)_*$, is surjective. This homomorphism is a homomorphism of polarized Hodge structures which then induces a surjective homomorphisms of the corresponding abelian varieties

$$J(\bar{W}^\natural) \oplus (\oplus_{t \in \mathbb{P}^1 \smallsetminus U} \mathrm{Pic}^0(X'_t)^\natural) \to J^3_{\mathrm{alg}}(\tilde{\mathscr{X}})^\natural\ .$$

Since $J^3_{\mathrm{alg}}(\mathscr{X})^\natural_{\mathbb{Q}} \simeq J(B)^\natural_{\mathbb{Q}}$, this finishes the proof of Theorem 9. $\square$

7. τ-action on $CH^2(S)$

Now we are ready to prove the identity action of the Nikulin involution on $CH^2(S)$ in case $d = 3$. The below proof is similar to the proof of Bloch's conjecture for finite quotients of quintics in $\mathbb{P}^3$ sketched in [21].

Theorem 10. *Let k be an algebraically closed field of characteristic 0, and let S be the intersection of a smooth cubic and a smooth quadric in $\mathbb{P}^4$ over k, both invariant under the involution τ changing the sign of two coordinates in the 4-dimensional projective space. Then the action $\tau^* : CH^2(S) \to CH^2(S)$ is the identity.*

Proof. As we work with Chow groups with coefficients in $\mathbb{Q}$, without loss of generality we may assume that $k = \mathbb{C}$. In the notation introduced in Section 4,

$$S = S_0 \ ,$$

where

$$S_0 = \mathscr{C}_0 \cap \mathscr{Q}_0$$

for smooth τ-invariant cubic $\mathscr{C}_0$ and quadric $\mathscr{Q}_0$ in $\mathbb{P}^4$. We need to show that the action of the involution τ on S induces the identity action on $CH^2(S)$.

The Chow group $CH^2(S)$ is a direct sum of $A^2(S)$ and $\mathbb{Q}$. As the action of τ does not change the degree of 0-cycles on S, in order to prove the identity action on $CH^2(S)$ it is enough to show that τ acts as the identity on $A^2(S)$. The latest group is generated by the differences of type $P - Q$, where P and Q are closed points on S. Thus, all we need to prove is that, for any two points P and Q on S, the 0-cycle $\tau(P) - \tau(Q)$ is rationally equivalent to the 0-cycle $P - Q$.

By Lemma 5, there exists another cubic

$$\mathscr{C}_1 \in \mathscr{L}_3 \smallsetminus \mathscr{M}_3 \ ,$$

such that P and Q sit on the surface $S_1 = \mathscr{C}_1 \cap \mathscr{Q}_0$. For short, let

$$\mathscr{Q} = \mathscr{Q}_0 \ .$$

Consider a pencil

$$f : \mathscr{Q} \dashrightarrow \mathbb{P}^1$$

of $K3$-surfaces obtained by restricting the pencil of cubics $|\mathscr{C}_t|_{t \in \mathbb{P}^1}$ passing through $\mathscr{C}_0$ and $\mathscr{C}_1$ on the quadric $\mathscr{Q}$. By Bertini's theorem, we may well assume that the base locus of the pencil f is a smooth irreducible curve

$$B = S_0 \cap S_1 \ .$$

Since P and Q sit on S_0 and S_1, they both are points on the curve B.

For any $t \in \mathbb{P}^1$ let

$$S_t = \mathscr{C}_t \cap \mathscr{Q}$$

and let

$$i_t : S_t \hookrightarrow \mathscr{Q}$$

be the corresponding closed embedding. Similarly like in Section 5, let

$$\tilde{f} : \tilde{\mathscr{Q}} \longrightarrow \mathbb{P}^1$$

be a resolution of the indeterminacy locus B. Let then

$$\tilde{i}_t : S_t \hookrightarrow \tilde{\mathscr{Q}}$$

be a closed embedding of S_t into $\tilde{\mathscr{Q}}$. If

$$S'_t \longrightarrow S_t$$

is a desingularization of S_t (the identity if S_t is smooth), let then

$$i'_t : S'_t \hookrightarrow \mathscr{Q} \qquad \text{and} \qquad \tilde{i}'_t : S'_t \hookrightarrow \tilde{\mathscr{Q}}$$

be compositions of i_t and $\tilde{i}_t$ with that desingularization. Finally, let

$$j_t : B \hookrightarrow S_t$$

be a closed embedding of the base locus into the fibre S_t, and let

$$j'_t : B \hookrightarrow S'_t$$

be the embedding of B into the desingularization S'_t of the fibre S_t (if S_t is smooth then $S'_t = S_t$). As $S = S_0$, it is natural to write $i = i_0$, $j = j_0$, etc.

The zero-cycle class $[P - Q] \in A^2(S)$ is in the image of the push-forward homomorphism

$$j_* : A^1(B) \longrightarrow A^2(S) \ .$$

Let $\beta \in A^1(B)$ be a cycle class such that

$$j_*(\beta) = [P - Q] \ .$$

Linking the current notation to those introduced in Section 2, if $G = \{\mathrm{id}, \tau\}$ is a group of order 2 generated by the involution τ, the group $A^1(B)$ is generated by two subgroups $A^1(B)^G$ and $A^1(B)^\sharp$. Then there exist two elements $\beta' \in A^1(B)^G$ and $\beta'' \in A^1(B)^\sharp$, such that

$$\beta = \beta' + \beta'' \ .$$

All we need is to show that $(j_0)_*(\beta'')$ vanishes.

In terms of Section 5, when $\mathscr{X}$ is $\mathscr{Q}$, all the Assumptions A, B, C and D hold true. By Theorem 9, the cycle class β'' is a sum of cycle classes of type $(j'_t)^*(\alpha')$, for $\alpha' \in A^1(S'_t)^\sharp$. For any such α' let α be the pull-back of α' to S_t. Then β'' is a sum of cycle classes of type $(j_t)^*(\alpha)$, for $\alpha \in A^1(S_t)^\sharp$. Notice that if S_t is singular, then j_t^* is nothing but the Gyzin homomorphism defined in §2.6 in [6].

For any $t \in \mathbb{P}^1$, such that $t \neq 0$, we have a Cartesian square

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "j_t"'] & S \arrow[d, "i"] \\ S_t \arrow[r, "i_t"] & \mathscr{Q} \end{tikzcd}$$

As .....[2], we have that

$$j_* j_t^* = i^* i_{t*} \ ,$$

as homomorphisms from $A^1(S_t)$ to $A^1(S)$, see [6], §6.2. Since

$$A^2(\mathscr{Q})^\natural = 0 \ ,$$

it follows that

$$j_* j_t^* = 0 \ .$$

[2]need to explain here. In particular, is i_t a regular embedding when S_t isn't smooth?

If $t = 0$ then, $j_*j^*(\alpha) = 0$ too. That can be deduced, for example, in the following way. The cycle class $j_*j^*(\alpha)$ is represented by an intersection of two divisors on the surface S. One of these two divisors is B, and another one is an algebraically trivial divisor D representing the class α. By the result in [2], the class of 0-cycle $D \cdot B$ must be, mod rational equivalence, proportional to the class of a point sitting on a (possibly singular) rational curve on S. As D is algebraically trivial, it follows that $[D \cdot B] = j_*j^*(\alpha) = 0$.

Thus, the cycle class β'', being a sum of cycle classes of type $(j_t)^*(\alpha)$, for $\alpha \in A^1(S_t)^\sharp$, goes to 0 under the push-forward homomorphism j_*, which finishes the proof of Theorem 10. □

8. τ-action on Prymians

Now we would like to go another way around and fibre a τ-invariant cubic $\mathscr{C} \subset \mathbb{P}^4$ by a pencil of τ-invariant quadrics in $\mathbb{P}^4$. In doing that we will understand the structure of the group $A^2(\mathscr{C})$, with regard to the action of the involution τ on it.

We will be now working over an algebraically closed subfield k in $\mathbb{C}$, such that $\mathrm{tr.deg}(\mathbb{C}/k) \neq 0$. Let $\mathscr{C}$ be a general smooth cubic in the linear series $\mathscr{L}$. In particular, $\mathscr{C}$ is invariant under the involution τ and contains the line l_τ. Restricting the projection of $\mathbb{P}^4$ onto Π_τ from the line l_τ onto the cubic $\mathscr{C}$, and blowing up the indeterminacy locus l_τ, we obtain a regular morphism $\hat{p}$ from the blow up $\hat{\mathscr{C}}$ onto Π_τ. The discriminant curve C consists of two smooth components C_2 and C_3 of degree 2 and 3 respectively, see Section 4.

For any point P on Π_τ the corresponding projecting plane

$$\Pi_P = \mathrm{span}(P, l_\tau)$$

intersects the cubic $\mathscr{C}$ along the line l_τ and a conic C_P. Since P, as well as the points of l_τ are fixed under the involution τ, the involution acts in the fibres of the projection p, which is good. If $P \in C = C_2 \cup C_3$ then the conic C_P splits into two lines,

$$C_P = l_P^+ \cup l_P^- \; .$$

Let

$$C_i' = \{l \in \mathrm{Gr}(2,5) \mid l \text{ is a line on } \mathscr{C}, \text{ such that } p\,(l \smallsetminus l_\tau) \in C_i\} = \\ = \{l \subset \mathscr{C} \mid \exists P \in C_i \text{ such that } l \in \{l_P^+, l_P^-\}\}\; ,$$

for $i = 2, 3$. Then C_2' and C_3' are two smooth irreducible algebraic curves which are double covers of the smooth irreducible curves C_2 and C_3 respectively,

$$C_2' \xrightarrow{2:1} C_2 \qquad \text{and} \qquad C_3' \xrightarrow{2:1} C_3 \; .$$

Moreover, these two double covers are ramified at the six points of intersection of C_2 and C_3. The Hurwitz formula shows that the genera of the curves C_2' and C_3' are 2 and 4 respectively,

$$g(C_2') = 2 \qquad \text{and} \qquad g(C_3') = 4 \; .$$

The following descriptions of the curves C_2' and C_3' mimic the classical setting when the discriminant curve C is irreducible, see [15].

Let

$$T\mathscr{C} \longrightarrow \mathscr{C}$$

be the tangent bundle of the cubic $\mathscr{C}$, and let

$$T = T\mathscr{C}|_{l_\tau}$$

be the restriction of $T\mathscr{C}$ on the line l_τ. Let also

$$\mathscr{V} = \mathbb{P}(T) \longrightarrow l_\tau$$

be the projectivization of the 3-dimensional vector bundle T over l_τ. Then, for any $D \in l_\tau$ let $D^\dagger$ be the point corresponding to the line l_τ in the fibre $\mathscr{V}_D$ of the morphisms $\mathscr{V} \to l_\tau$ over D. When D moves along the line l_τ, the point $D^\dagger$ moves along an algebraic curve

$$l_\tau^\dagger \subset \mathscr{V} \ ,$$

biregular to the line l_τ.

Next, for any point D on l_τ we have at most 6 lines passing through D on $\mathscr{C}$, and for all but finitely many points D on l_τ we have exactly 6 lines passing through D on $\mathscr{C}$. One of those 6 lines is, of course, the line l_τ itself. Let $D_i^\dagger$, $i = \overline{1,5}$ be the other 5 points in $\mathscr{V}_D$. When D runs over the line l_τ, the points $D_i^\dagger$ sweep up an algebraic curve $C^\dagger$ in $\mathscr{V}$,

$$C^\dagger \subset \mathscr{V} \ ,$$

which consists of two components biregular to the double covers C_2' and C_3',

$$C^\dagger = C_2^\dagger \cup C_3^\dagger \ ,$$

$$C_2^\dagger \simeq C_2' \qquad \text{and} \qquad C_3^\dagger \simeq C_3' \ .$$

Let now

$$J_i = J(C_i) \qquad \text{and} \qquad J_i^\dagger = J(C_i^\dagger)$$

be the Jacobians of the curves C_i and $C_i^\dagger$ respectively, where $i \in \{2,3\}$. The double covers $C_2^\dagger$ and $C_3^\dagger$ over the curves C_2 and C_3 respectively give rise to involutions

$$\iota_i : C_i^\dagger \longrightarrow C_i^\dagger \ ,$$

and, respectively, involutions on the Jacobians

$$\iota_i^* : J_i^\dagger \longrightarrow J_i^\dagger \ ,$$

$i \in \{2,3\}$. Let

$$\mathscr{P}_i = \mathrm{Prym}(C_i^\dagger/C_i) = \{P \in J_i^\dagger \mid \iota_i^*(P) = -P\}^0$$

be the connected component containing the 0 in the corresponding commutative group subvariety in $J_i^\dagger$, for each index $i = 2,3$. These are the Prym varieties for the double covers $C_2^\dagger/C_2$ and $C_3^\dagger/C_3$ respectively. Since the genus of C_2 is zero, the Prym variety $\mathscr{P}_2$ coincides with the whole Jacobian $J_2^\dagger$, which then is an abelian surface over k_0.

The involution τ on the cubic $\mathscr{C}$ induces two involutions, τ_2 on $C_2^\dagger$ and τ_3 on $C_3^\dagger$. Respectively, we have two involutions on the Jacobians,

$$\tau_2^* : J_2^\dagger \to J_2^\dagger \qquad \text{and} \qquad \tau_3^* : J_3^\dagger \to J_3^\dagger \ .$$

Theorem 11. *The involution τ_2^* coincides with the involution ι_2^* on $J_2^\dagger$, while the involution τ_3^* is the identity on $J_3^\dagger$.*

Proof. Let P be a point on the plane Π_τ, and let Π_P be the span of the point P and the line l_τ. Look at the equation of the cubic $\mathscr{C}$,

$$l_{00}(x_2,x_3,x_4)x_0^2+l_{11}(x_2,x_3,x_4)x_1^2+l_{01}(x_2,x_3,x_4)x_0x_1+f_3(x_2,x_3,x_4)=0\ ,$$

see the formula (4). Under an appropriate change of the coordinates x_3 and x_4, keeping the coordinates x_0, x_1 and x_2 untouched, the plane Π_P will be given by the equation

$$\Pi_P : x_3 = x_4 = 0\ .$$

Herewith, as the coordinates x_0, x_1 and x_2 remain the same, the involution τ in $\mathbb{P}^4$ can be expressed by the same formula 1, so that the equations 2 for l_τ and Π_τ remain the same too. Substituting $x_3 = x_4 = 0$ into the above equation for the cubic $\mathscr{C}$, we obtain the equation for the fibre $\Pi\cap\mathscr{C}$ of the projection $p:\mathscr{C}\dashrightarrow\Pi_\tau$ over the point

$$P=(0:0:1:0:0)$$

of the intersection of two planes Π_P and Π_τ. Namely,

$$\Pi_P\cap\mathscr{C} : x_2(\alpha x_0^2+\beta x_1^2+\gamma x_0x_1+\delta x_2^2)=0\ ,$$

where α, β, γ and δ are some numbers from k_0. If a point $A=(a_0:a_1:a_2:a_3:a_4)$ sits in $\Pi_P\cap\mathscr{C}$ and $a_2=0$ then A sits on the line l_τ. As we are interested in the fibre of the projection from $\mathscr{C}\smallsetminus l_\tau$ we must set $x_2\neq 0$. Then, if

$$E_P$$

is the Zariski closure of the set $(\Pi_P\cap\mathscr{C})\smallsetminus l_\tau$ in $\mathscr{C}$ the curve E_P is a conic defined by the equation

$$E_P : \alpha x_0^2+\beta x_1^2+\gamma x_0x_1+\delta x_2^2=0$$

in Π_P. Then

$$P\in C \Leftrightarrow \text{the conic } E_P \text{ splits in two transversal lines,}$$

$$E_P=l_P^+\cup l_P^-\ .$$

Moreover,

$$P\in C_3\Leftrightarrow\delta=0 \qquad\text{and}\qquad P\in C_2\smallsetminus C_3\Leftrightarrow\delta\neq 0\ .$$

Then we see that, if $P\in C_3$, the lines l_P^+ and l_P^- pierces the plane Π_τ at the point P. It follows then that $\tau_3^*=\mathrm{id}$.

Suppose $P\in C_2^\dagger$. Since E_P splits,

$$\alpha x_0^2+\beta x_1^2+\gamma x_0x_1+\delta x_2^2=\delta(x_2+b_0x_0+b_1x_1)(x_2-b_0x_0-b_1x_1)\ ,$$

so that the lines of E_P are defined by the equations

$$l_P^+ : x_2+b_0x_0+b_1x_1=0 \qquad\text{and}\qquad l_P^- : x_2-b_0x_0-b_1x_1=0\ .$$

Now we see that $\tau(l_P^+)=l_P^-$ and $\tau(l_P^-)=l_P^+$. □

Recall that $\mathscr{K}=\mathrm{cone}(l_\tau,C_2)$ is the quadric cone with vertex l_τ over the cubic C_2. This quadric $\mathscr{K}$ is τ-invariant as the line l_τ, as well as the conic C_2, both consist of fixed points of the involution τ.

Let $\mathscr{Q}_1$ be any smooth quadric in $\mathscr{L}_2$ and consider a general pencil $|\mathscr{Q}_t|_{t\in\mathbb{P}^1}$ through $\mathscr{K}$ and $\mathscr{Q}_1$. Intersecting the cubic $\mathscr{C}$ with the pencil $|\mathscr{Q}_t|_{t\in\mathbb{P}^1}$ we get a pencil of $K3$-surfaces,

$$q:\mathscr{C}\dashrightarrow\mathbb{P}^1\ ,$$

on the cubic threefold $\mathscr{C}$. Let $\mathscr{Q}_0$ be yet another smooth quadric in the pencil $|\mathscr{Q}_t|_{t\in\mathbb{P}^1}$, different from $\mathscr{Q}_1$. Then this pencil is generated by two τ-invariant smooth quadrics $\mathscr{Q}_0$ and $\mathscr{Q}_1$, so that we are in the situation described in Proposition 8. For any $t \in \mathbb{P}^1$ we write

$$S_t = \mathscr{Q}_t \cap \mathscr{C} \; .$$

Let now

$$Z = \mathscr{Q}_0 \cap \mathscr{Q}_1 \cap \mathscr{C}$$

be the base locus of the above pencil q. As the quadric $\mathscr{K}$ is singular, the curve Z is singular too, see Proposition 8. We now blow up the cubic $\mathscr{C}$ at Z,

$$\begin{tikzcd} \tilde{\mathscr{C}} \arrow[r, "s"] \arrow[dr, "\tilde{q}"'] & \mathscr{C} \arrow[d, dashed, "q"] \\ & \mathbb{P}^1 \end{tikzcd}$$

Let

$$K = k(\mathbb{P}^1)$$

be the function field on $\mathbb{P}^1$,

$$\eta = \mathrm{Spec}(K)$$

the generic point on $\mathbb{P}^1$ over k, and let

$$X = \tilde{\mathscr{C}}_\eta$$

be the generic fibre of the morphisms $\tilde{q}$.

Having embedded K into $\mathbb{C}$ by sending the transcendental parameter in to a transcendental number over k, one can consider also the $K3$-surface $X_{\mathbb{C}}$ over $\mathbb{C}$. Certainly, different embeddings of K into $\mathbb{C}$ will give different $K3$-surfaces over $\mathbb{C}$. Therefore, we fix an an embedding $K \hookrightarrow \mathbb{C}$ and let $\bar{K}$ be an algebraic closure of the field K in $\mathbb{C}$. Then we have that $X_{\bar{K}}$ is a $K3$-surface over $\bar{K}$, and $X_{\mathbb{C}}$ is obtained from $X_{\bar{K}}$ by the extension of scalars from $\bar{K}$ to $\mathbb{C}$. The above involution τ on $\mathscr{C}$ induces an involution

$$\tilde{\tau} : \tilde{\mathscr{C}} \longrightarrow \tilde{\mathscr{C}} \; ,$$

which, in one's turn, induces an involution

$$\tau : X \longrightarrow X$$

on the generic fibre. Then $\tau_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ induces the identity on $H^{0,2}(X_{\mathbb{C}})$, so that $\tau_{\mathbb{C}}$ is a Nikulin involution on $X_{\mathbb{C}}$, see [7]. This is why it makes sense to apply the reasoning from Section 3 to the surface X. By Theorem 10, the action $\tau^* : CH^2(X) \to CH^2(X)$ is the identity.

Notice that, due to Bloch and Srinivas, we know that the homological equivalence coincides with the algebraic equivalence for codimension 2 algebraic cycles on the cubic $\mathscr{C}$, see [3]. This is why, $CH^2(\mathscr{C})$ is isomorphic to a direct sum of the group $A^2(\mathscr{C})$ and a $\mathbb{Q}$-vector space V, such that $V \otimes_{\mathbb{Q}} \mathbb{Q}_l$ is isomorphic to a subspace in the 4-th Weil cohomology group $H^4_{\acute{e}t}(X_{\mathbb{C}}, \mathbb{Q}_l(2))$, for some prime l. Since the latest group is isomorphic to $\mathbb{Q}_l$, we have that $V \simeq \mathbb{Q}$, whence

$$CH^2(\mathscr{C}) \simeq A^2(\mathscr{C}) \oplus \mathbb{Q} \; .$$

The curve Z has ordinary double points. However, the closed embedding $Z \hookrightarrow \mathscr{C}$ is still local complete intersection, so it is regular. Consider the blow up Cartesian square in the category of schemes over k:

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{i}"] \arrow[d, "r"] & \tilde{\mathscr{C}} \arrow[d, "s"] \\ Z \arrow[r, "i"] & \mathscr{C} \end{tikzcd}$$

Here $\tilde{Z}$ is a proper transform of the curve Z under the blow up s, and $i : Z \hookrightarrow \mathscr{C}$ is a closed embedding of Z into $\mathscr{C}$. By Proposition 6.7 in [6], we have a short exact sequence of abelian groups

$$0 \to CH_1(Z) \xrightarrow{a} CH_1(\tilde{Z}) \oplus CH^2(\mathscr{C}) \xrightarrow{b} CH^2(\tilde{\mathscr{C}}) \to 0 \,,$$

where for any irreducible V of dimension d we write $CH_n(V) = CH^{d-n}(V)$. The group $CH_1(Z) = CH^0(Z)$ is obviously isomorphic to $\mathbb{Q}$, and τ acts on it identically.

Now look at the pull-back homomorphism

$$\Xi : CH^2(\tilde{\mathscr{C}}) \longrightarrow CH^2(X) \,,$$

where X is the generic fibre of the morphisms $\tilde{q} : \tilde{\mathscr{C}} \to \mathbb{P}^1$. This is a surjective homomorphism. Indeed, if Z is a closed integral subscheme in the scheme X over $K = k(t)$, and ζ its generic point, then we look at ζ as a point on the scheme $\tilde{\mathscr{C}}$ over k. Let $\mathscr{Z}$ be the closure of the point ζ on the scheme $\tilde{\mathscr{C}}$. Then $\Xi([\mathscr{Z}]) = [Z]$, where [ ] stays for the class of an integral sub-scheme in the Chow group. In other words, $\mathscr{Z}$ is a spread of Z in $\mathscr{C}$.

The diagram

$$\begin{tikzcd} CH^2(\tilde{\mathscr{C}}) \arrow[r, "\tilde{\tau}^*"] \arrow[d, "\Xi"] & CH^2(\tilde{\mathscr{C}}) \arrow[d, "\Xi"] \\ CH^2(X) \arrow[r, "\tau^*"] & CH^2(X) \end{tikzcd}$$

obviously commutes. It means that we can try to say something about the action τ^* at the foot of the diagram looking at action of the top homomorphism $\tilde{\tau}^*$.

Now, since $\tilde{Z} = \mathbb{P}(\mathscr{N})$, where $\mathscr{N}$ is the locally free normal sheaf of rank 2 of Z inside $\mathscr{C}$, the variety $\tilde{Z}$ is a ruled surface over the curve Z, so that

$$CH_1(\tilde{Z}) = CH^1(\tilde{Z}) \simeq \mathbb{Q} \oplus CH^1(Z) \,,$$

where the first summand is generated by a section of the canonical morphism $\tilde{Z} \to Z$.

Notice that, for each point P of the curve Z we paste into P a horizontal line $h_P = \mathbb{P}^1$ in $\mathscr{C}$, which is then a section of the regular morphism $\tilde{q}$. This line h_P gives rise to a k-rational point $P^\flat$ on the generic fibre X over k. When P runs Z, the point $P^\flat$ sweeps up a curve $Z^\flat$ in X. Since Z is irreducible and has ordinary

double points, the curve $Z^\flat$ is irreducible and has ordinary double points too. Moreover,

$$CH^1(Z) \simeq CH^1(Z^\flat)\ .$$

Combining this isomorphism with the obvious homomorphism $CH^1(Z) \to CH^1(\tilde{Z})$ we obtain a homomiorphism

$$c : CH^1(Z^\flat) \longrightarrow CH^1(\tilde{Z})\ .$$

Let

$$j^\flat : Z^\flat \hookrightarrow X$$

be the closed embedding of the curve $Z^\flat$ into X over k, and let

$$\tilde{j} : Z \hookrightarrow \tilde{\mathscr{C}}$$

be the closed embedding of the base locus Z into the cubic $\mathscr{C}$ over k_0. Then we obtain a commutative square of τ-equivariant homomorphisms

$$\begin{tikzcd} CH^1(Z^\flat) \arrow[r, "j^\flat_*"] \arrow[d, "c"] & CH^2(X) \\ CH^1(\tilde{Z}) \arrow[r, "\tilde{j}_*"] & CH^2(\tilde{\mathscr{C}}) \arrow[u, "\Xi"'] \end{tikzcd}$$

Notice that the composition $\Xi \circ \tilde{j}_*$ is the same as the composition

$$CH_1(\tilde{Z}) \hookrightarrow CH_1(\tilde{Z}) \oplus CH^2(\mathscr{C}) \xrightarrow{b} CH^2(\tilde{\mathscr{C}}) \xrightarrow{\Xi} CH^2(X)\ .$$

Let $\mathscr{P}_i$ be the Prym variety corresponding to the ι_i^*-action on the Jacobian $J_i^\dagger$, for $i = 2, 3$. Similarly to the result in [15], the Chow group $A^2(\mathscr{C})$ is isomorphic to the quotient-group

$$(\mathscr{P}_2 \oplus \mathscr{P}_3)/R \otimes \mathbb{Q}\ ,$$

where R is a subgroup coming from 6 points in the intersection of the curves C_2 and C_3. Let B_i be the subgroup in $A^2(\mathscr{C})$ corresponding to the Prymian $\mathscr{P}_i$ under the above isomorphism.

Theorem 12. *There Chow group $A^2(\mathscr{C})$ is generated by the two τ-equivariant subgroups B_2 and B_2 in it. The involution τ^* acts identically on B_3, τ^* acts as multiplication by -1 on B_2, and the image of the pull-back $s^*(B_2)$ under the homomorphism Ξ vanishes.*

Proof. We will be using the threefold $\mathscr{V}$ and the curves $l_\tau^\dagger$ and $C^\dagger = C_2^\dagger \cup C_3^\dagger$ in it. The curve $l_\tau^\dagger$ does not meet the curve $C^\dagger$ on $\mathscr{V}$. The curve $C_2^\dagger$ meets the curve $C_3^\dagger$ along 6 points. Let

$$\mathscr{C}' = \mathrm{Bl}_{C_3^\dagger \cup l_\tau^\dagger}(\mathscr{V})$$

be a blow up of $\mathscr{V}$ at $C_3^\dagger \cup l_\tau^\dagger$, let $C_2^{\dagger\prime}$ be the proper transform of $C_2^\dagger$ and let

$$\mathscr{C}'' = \mathrm{Bl}_{C_2^{\dagger\prime}}(\mathscr{C}')$$

be a blow up of $\mathscr{C}'$ at $C_2^{\dagger\prime}$. As in [15], one can construct a regular map

$$\phi : \mathscr{C}'' \longrightarrow \mathscr{C} \ ,$$

which will be generically $2 : 1$. As we work with Chow groups with coefficients in $\mathbb{Q}$, the pull-back

$$\phi^* : A^2(\mathscr{C}) \longrightarrow A^2(\mathscr{C}'')$$

is an injection. Proposition 13 in [18] shows that $A^2(\mathscr{C})$ is τ-equivariantly embedded into the product

$$J_2^\dagger \oplus J_3^\dagger \ .$$

By Theorem 11, the involution τ_2^* coincides with the involution ι_2^* on $J_2^\dagger$, while the involution τ_3^* is the identity on $J_3^\dagger$. It follows that the action of τ^* on B_2 is multiplication by -1, and the action of τ^* on B_3 is the identity.

Finally, we observe that the lines on $\mathscr{C}$, corresponding to the points on the curve $C_2^\dagger$, all lie in the intersection of the cubic $\mathscr{C}$ with the quadric K. Then the above inclusion $W^- \subset J_2^\dagger$ implies that $\Xi(W^-) = 0$, and the theorem in the introduction is proved for $L = K$.

□

Remark 13. Let $\mathrm{sp} : CH^2(X) \to CH^2(S)$ be a specialization homomorphism from the Chow group of the scheme-theoretic generic fibre X of the pencil $|S_t|_{t\in\mathbb{P}^1}$ to the Chow group of the geometric generic fibre $S = \mathscr{C} \cap \mathscr{Q}_0$ over k, see [6]. This specialization is onto. Indeed, let c_S be the class in $CH^2(S)$ generated by points sitting on rational curves on S, as in [2]. Take a point P on S. As P sits on the cubic $\mathscr{C}$, there is a line l on $\mathscr{C}$ passing through P. If this line is in S, then P belongs to the class c_S. Then the class of the point P is in the image of sp. If not, then the line l is horizontal with respect to the morphism $q : \mathscr{C} \dashrightarrow \mathbb{P}^1$. Then we have two possibilities – either l is tangent to S at P, and then the class of P is in the image of sp, or l intersects S transversally at P and yet another point P'. Repeat the procedure again, but with regard to the point P'. If a line l' passing through the point P' on $\mathscr{C}$ lies in S or is tangent to S, then again the class of P sits in the image of the map sp. If not, we draw a plane Π through the lines l and l' , which will intersect $\mathscr{C}$ along three horizontal lines l, l' and l''. These three lines give rise to three points on the generic fibres, which allow to construct a cycle class mapped in to the class of P under the specialization homomorphism sp.

Remark 14. The above arguments are over an arbitrary but fixed algebraically closed subfield k in $\mathbb{C}$. Suppose now that $k \subset l$ is a field extension, and l is algebraically closed. Then we also have a cubic $\mathscr{C}_l \subset \mathbb{P}^4_l$, whose intersection by the same pencil of quadrics, with scalars extended to l, is nothing but the rational map $q_l : \mathscr{C}_l \dashrightarrow \mathbb{P}^1_l$ over l. Let $L = K \otimes_k l = l(t)$ be the function field on $\mathbb{P}^1_l$, and let $\eta_l = \mathrm{Spec}(L)$ be the generic point on $\mathbb{P}^1_l$ over l. Then $X_L = (\mathscr{C}_l)_{\eta_l}$ is the generic fibre of the rational map $\mathscr{C}_l \dashrightarrow \mathbb{P}^1_l$. Applying the same arguments as above, but replacing now the ground field k by the field l, we get that the action $\tau_L^* : CH^2(X_L) \to CH^2(X_L)$ is the identity. In other words, our arguments are stable under scalar extensions.

Remark 15. The reader should not be confused about the pull-back homomorphism Ξ in the above reasoning. It is a surjection from a weakly representable

codimension 2 Chow group $A^2(\mathscr{C})$ for the cubic $\mathscr{C}$, which is a variety over the field k, onto the Chow group $A^2(X)$ of zero-cycles on the $K3$-surface X, which is a variety over the field $K = k(t)$. Therefore, weak representability of $A^2(\mathscr{C})$ does not imply, of course, weak representability of zero-cycles on X, which is not possible by Mumford's result.

9. Smooth base locus

Let again S be the intersection of any two generic τ-invariant cubic $\mathscr{C} \in \mathscr{L}_3$ and quadric $\mathscr{Q} \in \mathscr{L}_2$ in $\mathbb{P}^4$, all defined over a certain algebraically closed field k of zero characteristic. Let $\mathscr{Q}_0 = \mathscr{Q}$ and let $\mathscr{Q}_1$ yet another smooth τ invariant quadric in $\mathscr{L}_2$, such that the base locus Z of the restriction of the pencil $|\mathscr{Q}_t|_{t \in \mathbb{P}^1}$ through $\mathscr{Q}_0$ and $\mathscr{Q}_1$ on $\mathscr{C}$ is smooth. We wish now to study algebraic cycles on $S = \mathscr{C} \cap \mathscr{Q}$ supported on a curve Z.

Since $\mathscr{C} \cap \mathscr{Q}_0$ is a $K3$-surface, it's canonical class vanishes. Then, by the adjunction formula,

$$2g_Z - 2 = \deg(Z \cdot Z)\ .$$

Pick up another quadric $\mathscr{Q}_1'$ from the linear series of quadrics in $\mathbb{P}^4$. If $Z' = \mathscr{C} \cap \mathscr{Q}_0 \cap \mathscr{Q}_1'$ then

$$2g_Z - 2 = \deg(Z \cdot Z') = \deg(\mathscr{C} \cdot \mathscr{Q}_0 \cdot \mathscr{Q}_1 \cdot \mathscr{Q}_1') = 3 \cdot 2^3 = 24\ ,$$

whence

$$g_Z = 13\ .$$

In other words, the Hodge summand $H^{0,1}(Z)$ is of dimension 13.

Let now

$$j : Z \hookrightarrow S$$

be a closed embedding. Using some extra arguments one can prove the following corollary from Theorem 10:

Corollary 16. *The push-forward homomorphism*

$$j_* : A^1(Z) \longrightarrow A^2(S)$$

vanishes.

Proof. Let

$$i : Z \hookrightarrow \mathbb{P}^4$$

be a close embedding of Z into $\mathbb{P}^4$, let $\mathscr{O}_Z$ and Ω_Z be, respectively, the structure sheaf and the sheaf of differentials on the curve Z in $\mathbb{P}^4$. Consider a standard short exact sequence

$$0 \longrightarrow \mathscr{I}_Z \longrightarrow \mathscr{O} \longrightarrow i_*\mathscr{O}_Z \to 0\ ,$$

in the category of coherent sheaves on $\mathbb{P}^4$. Here $\mathscr{I}_Z$ is the sheaf of ideals of the closed subscheme Z in $\mathbb{P}^4$. Twisting by 2 yields a new short exact sequence

$$0 \longrightarrow \mathscr{I}_Z(2) \longrightarrow \mathscr{O}(2) \longrightarrow i_*\mathscr{O}_Z(2) \to 0\ .$$

Cohomology of coherent sheaves on $\mathbb{P}^4$ give rise to a long exact sequence of abelian groups

(8)

$$0 \to H^0(\mathbb{P}^4, \mathscr{I}_Z(2)) \to H^0(\mathbb{P}^4, \mathscr{O}(2)) \to H^0(\mathbb{P}^4, i_*\mathscr{O}_Z(2)) \to H^1(\mathbb{P}^4, \mathscr{I}_Z(2)) \to ...$$

Since

$$H^0(\mathbb{P}^4, i_*\mathscr{O}_Z(2)) = H^0(Z, \mathscr{O}_Z(2))$$

by Lemma 2.10 in [8], and the the canonical sheaf $\omega_Z = \Omega_Z$ is isomorphic $\mathscr{O}_Z(2)$ on Z,

$$\omega_Z \simeq \mathscr{O}_Z(2) \; ,$$

we obtain that

$$H^0(\mathbb{P}^4, i_*\mathscr{O}_Z(2)) \simeq H^0(Z, \Omega_Z) = H^{0,1}(Z) \; . \tag{9}$$

In other words, $H^0(\mathbb{P}^4, i_*\mathscr{O}_Z(2))$ is just another expression of the substantial part in the second cohomology of Z.

Let now $x = (x_0 : x_1 : x_2 : x_3 : x_4)$ be the coordinates in $\mathbb{P}^4$. Let then

$$\phi = \phi(x)$$

be the cubic form defining $\mathscr{C}$, and let

$$f_0 = f_0(x) \qquad \text{and} \qquad f_1 = f_1(x)$$

be the forms defining the quadrics $\mathscr{Q}_0$ and $\mathscr{Q}_1$ respectively. If

$$E = \mathscr{O}(2) \oplus \mathscr{O}(2) \oplus \mathscr{O}(3) \; ,$$

the triple

$$(f_0, f_1, \phi)$$

defines a global section

$$s \in H^0(\mathbb{P}^4, E) = H^0(\mathbb{P}^4, \mathscr{O}(2)) \oplus H^0(\mathbb{P}^4, \mathscr{O}(2)) \oplus H^0(\mathbb{P}^4, \mathscr{O}(3)) \; .$$

Dualizing the corresponding morphism $s : \mathscr{O} \to E$ we obtain a new morphism of sheaves

$$\check{s} : \check{E} = \mathscr{O}(-2) \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3) \longrightarrow \mathscr{O} \; ,$$

such that

$$\mathrm{coker}(\check{s}) = i_*\mathscr{O}_Z \; ,$$

because the scheme of zeros of the section s is exactly the curve Z. Since Z is a complete intersection of three hypersurfaces in $\mathbb{P}^4$, the exact sequence

$$\check{E} \to \mathscr{O} \to i_*\mathscr{O}_Z \to 0$$

extends to the Koszul resolution

$$0 \to \wedge^3\check{E} \to \wedge^2\check{E} \to \check{E} \to \mathscr{O} \to i_*\mathscr{O}_Z \to 0 \; .$$

Since $\check{E} = \mathscr{O}(-2) \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3)$, we have that

$$\wedge^2\check{E} = \mathscr{O}(-4) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-5)$$

and

$$\wedge^3\check{E} = \mathscr{O}(-2) \otimes \mathscr{O}(-2) \otimes \mathscr{O}(-3) = \mathscr{O}(-7) \; .$$

Twisting everything by 2 we obtain an exact sequence

$$0 \to \mathscr{O}(-5) \to \mathscr{O}(-2) \oplus \mathscr{O}(-3) \oplus \mathscr{O}(-3) \to \mathscr{O} \oplus \mathscr{O} \oplus \mathscr{O}(-1) \to \mathscr{O}(2) \to i_*\mathscr{O}_Z(2) \to 0 \; .$$

Let's saw up it into three short exact ones:

$$0 \to \mathscr{O}(-5) \to \mathscr{O}(-2) \oplus \mathscr{O}(-3) \oplus \mathscr{O}(-3) \to A \to 0 \ , \tag{10}$$

$$0 \to A \to \mathscr{O} \oplus \mathscr{O} \oplus \mathscr{O}(-1) \to B \to 0 \tag{11}$$

and

$$0 \to B \to \mathscr{O}(2) \to i_* \mathscr{O}_Z(2) \to 0 \ . \tag{12}$$

Notice that from the last sequence we get

$$B \simeq \mathscr{I}_Z(2) \ .$$

Since

$$H^i(\mathbb{P}^4, \mathscr{O}(n)) = 0$$

for all integers n and all $0 < i < 4$, and

$$H^0(\mathbb{P}^4, \mathscr{O}(n)) = 0 \ ,$$

if $n < 0$, see Theorem 5.1 in [8], from the sequence (10) we have that

$$H^i(\mathbb{P}^4, A) = 0$$

for $i = 0, 1, 2$. Then, from (11) we get

$$H^1(\mathbb{P}^4, B) \simeq H^1(\mathbb{P}^4, \mathscr{I}_Z(2)) = 0 \ .$$

In view of (8) and (9) we get a short exact sequence

$$0 \to H^0(\mathbb{P}^4, \mathscr{I}_Z(2)) \to H^0(\mathbb{P}^4, \mathscr{O}(2)) \to H^{0,1}(Z) \to 0 \ .$$

Since $H^0(\mathbb{P}^4, A)$ and $H^1(\mathbb{P}^4, A)$ vanish, $H^0(\mathbb{P}^4, \mathscr{O}) = k$ and $H^0(\mathbb{P}^4, \mathscr{O}(-1))$ vanishes too, from (11) we get

$$H^0(\mathbb{P}^4, \mathscr{I}_Z(2)) \simeq H^0(\mathbb{P}^4, B) \simeq k \oplus k \ .$$

Let V be a 5-dimensional k-vector space, such that

$$\mathbb{P}^4 = \mathbb{P}(V) \ .$$

Then one has a canonical isomorphism

$$H^0(\mathbb{P}^4, \mathscr{O}(2)) \simeq \mathrm{Sym}^2(\check{V}) \ ,$$

where $\mathrm{Sym}^2 \check{V}$ is the symmetric square of the dual space $\check{V} = \mathrm{Hom}_k(V, k)$, which can be identified with the space of quadratic forms on V. In particular,

$$\dim_k H^0(\mathbb{P}^4, \mathscr{O}(2)) = \binom{6}{2} = 15 \ ,$$

so that

$$\dim H^{0,1}(Z) = 15 - 2 = 13 \ ,$$

as above.

Let now

$$V = V^- \oplus V^+$$

be a splitting of the space V into two subspaces, such that V^- corresponds to the coordinates x_0 and x_1, and V^+ corresponds to the coordinates x_2, x_3 and x_4. Then

$$l_\tau \simeq \mathbb{P}(V^-) \qquad \text{and} \qquad \Pi_\tau \simeq \mathbb{P}(V^+)\ .$$

We have that

$$\mathrm{Sym}^2(\check{V}) \simeq \mathrm{Sym}^2(\check{V}^- \oplus \check{V}^+) \simeq \mathrm{Sym}^2(\check{V}^-) \oplus (\check{V}^- \otimes \check{V}^+) \oplus \mathrm{Sym}^2(\check{V}^+)\ ,$$

getting

$$H^0(\mathbb{P}^4, \mathscr{O}(2)) \simeq \mathrm{Sym}^2(\check{V}^-) \oplus (\check{V}^- \otimes \check{V}^+) \oplus \mathrm{Sym}^2(\check{V}^+)\ .$$

As result, we obtain the following short exact sequence

$$0 \to k_0 \oplus k_0 \to \mathrm{Sym}^2(\check{V}^-) \oplus (\check{V}^- \otimes \check{V}^+) \oplus \mathrm{Sym}^2(\check{V}^+) \to H^{0,1}(Z) \to 0\ .$$

We will be using now this sequence to understand the action of the involution τ in the Hodge cohomology $H^{0,1}(Z)$.

The image of $k \oplus k$ in $\mathrm{Sym}^2(\check{V}^-) \oplus (\check{V}^- \otimes \check{V}^+) \oplus \mathrm{Sym}^2(\check{V}^+)$ is generated by the quadratic forms f_0 and f_1. Since f_0 and f_1 are supposed to be τ-invariant, this image is also τ-invariant. Besides both quadratic forms f_0 and f_1 lie in the τ-invariant subspace

$$\mathrm{Sym}^2(\check{V}^-) \oplus \mathrm{Sym}^2(\check{V}^+)$$

in $H^0(\mathbb{P}^4, \mathscr{O}(2))$, and they are both have non-trivial components in $\mathrm{Sym}^2(\check{V}^-)$ and in $\mathrm{Sym}^2(\check{V}^+)$. The tensor product

$$\check{V}^- \otimes \check{V}^+$$

is anti-invariant under the action of τ, i.e. τ acts by multiplication by -1 on it. Since

$$\dim(\mathrm{Sym}^2(\check{V}^-) \oplus \mathrm{Sym}^2(\check{V}^+)) = 9$$

and

$$\dim(\check{V}^- \otimes \check{V}^+) = 6\ ,$$

we deduce that $H^{0,1}(Z)$ splits in to its τ-invariant and τ-anti-invariant blocks,

$$H^{0,1}(Z) = H^{0,1}(Z)_+ \oplus H^{0,1}(Z)_-\ ,$$

where

$$\dim(H^{0,1}(Z)_+) = 7 \qquad \text{and} \qquad \dim(H^{0,1}(Z)_-) = 6\ .$$

This computation shows that the Jacobian J of the curve Z, up to isogeny, splits into two non-trivial components, one of which is anti-invariant with respect to the action of the involution τ. As it was explained in [21], the kernel of j_* can be a countable union of translates of a certain $\mathbb{Q}$-vector space $A \otimes \mathbb{Q}$, where A is an abelian subvariety in the Jacobian J. Moreover, this A can be either 0 or the whole Jacobian J, otherwise it would lead to a contradiction with irreducibility of monodromy action on cohomology of Z, loc.cit. If $A = 0$ then the kernel of j_* is countable. But the above computation then shows that the anti-invariant part of the Jacobian is mapped into the invariant Chow-group $A^2(S)$ with a countable kernel. This is an obvious contradiction, as the ground field k is assumed to be not countable. Therefore, $A = J$ and j_* vanishes. □

10. Concluding remarks

Let S be an intersection of τ-invariant cubic and quadric from the linear series $\mathscr{L}_3$ and $\mathscr{L}_2$ respectively. Let also $T = S/\tau$ be the quotient of S by the involution τ and let $T' \to T$ be a resolution of eight nodal singularities on T coming from eight fixed points of the involution τ on the $K3$-surface S. Then T' is again a $K3$-surface, and as we have already seen in Section 3, the transcendental motive $t^2(T')$ of T' is isomorphic to the transcendental motive $t^2(S)$ of the original $K3$-surface S. A natural question is now whether the above results can help us to prove that the motive of the surface S is finite-dimensional? In other words, whether the motive $t^2(T')$ is simpler, in this or that sense, that the motive $t^2(S)$?

Let $\mathscr{B}$ be a family of lines in $\mathbb{P}^4$ each of which intersects the line l_τ at one point and the plane Π_τ at one point. Naturally, $\mathscr{B} \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Here the isomorphism is provided by sending a line l_b, corresponding to a point b in $\mathscr{B}$, to an ordered pair (P, Q), where $P = l_b \cap l_\tau$ and $Q = l_b \cap \Pi_\tau$.

Notice that for any point R in $\mathbb{P}^4 \smallsetminus \{l_\tau \cup \Pi_\tau\}$ there exists a unique line l_R in the family $\mathscr{B}$ passing through R. Then we have a regular map from $\mathbb{P}^4 \smallsetminus \{l_\tau \cup \Pi_\tau\}$ to $\mathscr{B}$, sending R to l_R, and the corresponding rational map $g : \mathbb{P}^4 \dashrightarrow \mathscr{B}$. Let $\tilde{g} : \tilde{\mathbb{P}}^4 \dashrightarrow \mathscr{B}$ be the resolution of the indeterminacy of the rational map g. For any point $b \in \mathscr{B}$ the pre-image $\tilde{g}^{-1}(b)$ is the whole line l_b. Let $\tilde{S} \subset \tilde{\mathbb{P}}^4$ be Zariski closure of the quasi-projective surface $S \smallsetminus \{l_\tau \cup \Pi_\tau\}$ in $\tilde{\mathbb{P}}^4$. Then $\tilde{S} \simeq S$, and we have a regular map $\tilde{g}|_{\tilde{S}} : \tilde{S} \longrightarrow \mathscr{B}$. Let l_b be a line in $\mathscr{B}$, such that the intersection of l_b and S is non-empty. Since S lies in a quadric $\mathscr{Q}$, the intersection $S \cap l_b$ can be either two points P and P' of intersection multiplicity 1, such that $P' = \tau(P)$, or one point P of intersection multiplicity 2, such that $\tau(P) = P$. If now $\tilde{T}$ is the image of the regular map $\tilde{g}|_{\tilde{S}}$ from $\tilde{S}$ to $\mathscr{B}$, then $\tilde{g}|_{\tilde{S}} : \tilde{S} \to \tilde{T}$ is generically $2:1$ and $\tilde{T}$ can be identified with the surface T.

Let now $r : \tilde{T} \to \mathbb{P}^2$ be a composition of the closed embedding of $\tilde{T}$ into $\mathscr{B} \simeq \mathbb{P}^1 \times \mathbb{P}^2$ with the projection onto the second factor $\mathbb{P}^2$. Since the cubic $\mathscr{C}$ is defined by an equation

$$l_{00}(x_2, x_3, x_4)x_0^2 + l_{11}(x_2, x_3, x_4)x_1^2 + l_{01}(x_2, x_3, x_4)x_0x_1 + f_3(x_2, x_3, x_4) = 0 \ ,$$

and the quadric $\mathscr{Q}$ is defined by an equation

$$\alpha_{00}x_0^2 + \alpha_{11}x_1^2 + \alpha_{01}x_0x_1 + f_2(x_2, x_3, x_4) = 0$$

(see Section 4), the singular surface $\tilde{T}$ is defined by the equation

$$\left(l_{00}(x_2, x_3, x_4)x_0^2 + l_{11}(x_2, x_3, x_4)x_1^2 + l_{01}(x_2, x_3, x_4)x_0x_1\right) f_2(x_2, x_3, x_4) -$$
$$\left(\alpha_{00}x_0^2 + \alpha_{11}x_1^2 + \alpha_{01}x_0x_1\right) f_3(x_2, x_3, x_4) = 0 \ ,$$

which can be viewed as a quadric equation on two variables x_0 and x_1, whose coefficients are cubic forms in variables x_2, x_3 and x_4. Then we see that the discriminant of this quadric equation will be a form of degree 6. It means that the brunching locus of the regular map $\tilde{S} \xrightarrow{2:1} \tilde{T}$ is a curve of degree 6 of genus 2 in $\mathbb{P}^1 \times \mathbb{P}^2$. As T' is just a blow up of the surface $T \simeq \tilde{T}$, we have a pretty

explicit geometrical description of T'. Yet, it does not help to understand the nature of the motive of the surface T', and new methods are needed here.

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